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COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

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THE VALUE OF THE NON-ATOMIC GAME ARISING FROM A RATE-SETTING APPLICATION AND RELATED PROBLEMS

by

(10) Joseph Raanan

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Abstract

The work is motivated by the following problem: bulk-service telephone lines were installed at Cornell University, to be used for long-distance calls. The charges paid to the telephone company are mostly fixed monthly charges and are not usage-related. The problem is how to allocate these costs back to the users in a per call fashion, and how to do it in a way that is fair and efficient. The problem was solved by using the value of the associated non-atomic game. To be able to do this, the theory of non-atomic games had to be extended by weakening certain differentiability requirements. This is done here; in addition a number of results about full-range game are obtained.

Next the problem of non-atomic linear production games is studied. A number of results about the cores of such games are obtained, extending and strengthening similar results about finite linear production games. In addition, some results about the value of such games are established, and relationships between the core and the value are derived for a special case.

CHAPTER I

INTRODUCTION

1.1 Prologue

The motivation for this work derives from a problem of setting rates for long-distance telephone calls placed by members of the Cornell Community. A solution based on the value of an associated non-atomic game was proposed and adopted. However, since the theory of non-atomic games as it existed at that time did not cover exactly the application, it had to be extended slightly. In the process of extending the known results to the rate-setting problem on hand, some other results about value of certain non-atomic games were obtained, namely those about full-range games.

Next, a closely related problem was studied--that of non-atomic linear production games. This problem is directly related to our rate-setting problem since the latter can be viewed as a non-atomic production game, albeit not a linear one. However, starting with the linear case may lead to results in the non-linear case. Extending the ideas of linear production game from finitely many players to the non-atomic case provided some interesting results about cores, values and equilibrium prices in the non-atomic case that do not hold in the finite case.

The next section presents the rate-setting problem that motivated this work. Chapter II serves a dual purpose; most of the definitions and notations are presented there (except those that are too specific--these will be introduced as needed), and it gives the results about values of full-range non-atomic games. Chapter III deals with the value on the space $\overline{GDD}$, the space generated by games whose structure is the same as that of our rate-setting game. In Chapter IV we make the connection between the problem and the

theory developed in Chapter III, and show how, indeed, the original problem is solved.

Chapter V deals with non-atomic linear production games. Linear production games were introduced by Owen [22]; here we generalize those results to the non-atomic case, and find some relationships between the cores of such games and their values.

The application of non-atomic game theory to solve an actual, "real-world" problem was, to the best of our knowledge, never done before. While the use of game theory for the determination of rates or costs is not new (see [3], [4], [5], [8], [10], [11], [12], [13], [14], [16], [17], [19], [25] and [26]) this seems to be the first case of using non-atomic game theory to solve the problems of a real client.

Moreover, the theory of non-atomic games was developed mainly as an approximation for very large--but *finite*--games. It affords both the use of the tools of mathematical analysis as well as a feasible computational method. (The computation of the value for a general n-person game would be practically infeasible even on a computer for any n moderately large.) Our problem, however, is one where the continuous nature of the game is inherent--the players are instants of telephone calls which can best be modeled using a continuous model.

As a final note we wish to say that the methods of setting rates described here and in [2] should be applicable to a number of other utilities such as water, steam and electricity where the units of service are continuous and the costs involve both high initial set-up costs and non-linear variable costs. A study, aimed at trying to apply these methods to electric utility rates is being contemplated at this time and may appear later.

1.2 The Rate-Setting Problem

In addition to Direct Distance Dialing (DDD) there are at least two more ways of obtaining long-distance telephone service, both affording some savings over DDD if used judiciously.

The first of these are the Foreign Exchange (FX) lines. Under this arrangement, the user has a phone (or a line) connecting him with an exchange in another city and state. The second of these money-saving devices are the Wide Area Telecommunication Service (WATS) lines. The area of the continental U.S. outside the user's own state is divided into 5 concentric bands, with band $i+1$ containing band i, $i = 1,2,3,4$. Band 1 usually contains just the neighboring states, while band 5 contains all of the continental U.S. (outside the user's own state). WATS lines servicing these bands can be obtained from the telephone company.

The costs associated with each of these 3 different ways of obtaining long-distance telephone service are as follows:

1. A call using DDD is charged, basically, by its length and by the distance to its destination.

2. The cost of an FX line is composed of two parts: One is the fixed monthly charge for the line between the user and the Foreign Exchange, and the other is the regular charge for telephone service at that Foreign Exchange. The line-rental charge is distance-dependent but it also involves some fixed (distance-independent) charges for connecting equipment; the charge for the local service at the Foreign Exchange usually consists of message-units (timed or untimed--depending on the locality).

3. WATS lines are available, for each of the 5 bands, under two different rate structures. Under both rate structures the user is not charged for individual calls but rather for the total time in the month during which

the line was in use. There is no differentiation by such factors as the duration of an individual call or the exact destination within the band of any given call. Under one structure, the initial monthly fee covers 240 hours of use with an additional charge for every hour above this limit. The other offers only 10 hours for the initial fee (which is smaller than under the first plan), but charges higher incremental rates for use exceeding this 10 hour limit.

The first question that arises after all this data is available is: Which, if any, of these services should be bought? This question was addressed by Lampell in [9]. There, a queueing model of a telephone system was built and a computer code was written that generates the optimal (least-cost) line configuration for a given demand, using a branch and bound algorithm. If the demand is, indeed, not high enough to justify any WATS or FX lines, the solution will be the 0 configuration. (DDD lines are not included in this procedure since these are always available as part of the basic service, without any additional cost.)

The telephone system at Cornell includes a computerized device for handling the outgoing calls. It provides Least-Cost Routing, meaning that it is programmed to route each call onto the best sequence of FX or WATS lines. For example, if the call coming into the device has Washington, D.C. as its destination, the device will first try to place the call on a Washington, D.C. FX line. If all these lines are busy, WATS 2 lines will be tried next, followed by WATS 3, WATS 4 and WATS 5--in that order. If all of these lines are busy the call will be sent DDD. In addition to Least-Cost Routing, the device also provides security checks to prevent unauthorized users from placing a call, and detailed bookkeeping information--who called, where to, how long the call lasted, etc.

Once the question of selecting the optimal line configuration that will best serve a given load was settled, a second question arose: How are the charges to be allocated to the users? While with DDD every long-distance phone call is charged back to the user who placed it, no such easy way of charging the users exists here, since most charges are fixed--line rental, device rental and maintenance, operator's salary--and are not directly associated with any individual call. It is this question--to be precisely stated shortly--that motivated this work.

At first, it seems that some of the costs incurred can be directly attributed to individual calls. For example, message units for calls using FX lines or the incremental charges (on WATS lines), to calls that cause these charges are such costs. However, objections are immediately raised even to those claims. For the first case, the question of allocating overhead expenses remains unsolved while for the second, the objection is even simpler: since the timing of calls accumulates on a monthly basis, all the incremental charges will be incurred by calls placed later in the month. But, is it indeed right to charge those calls differently just because they happen to be placed late rather than early in the month? The answer seems to be no.

We therefore had to look for a solution to this problem that would satisfy the following requirements:

A. The solution should be in the form of rates, i.e., charge per minute.

B. The revenue produced by these rates must exactly cover the expenses of operating the system and providing the service.

C. The rates must be "fair", or "symmetric", since calls are charged to different accounts and budgets (such as Research Grants, administrative

funds, etc.), and all must be charged the same. That is, in short, two calls made to the same destination during the same period in the day must be charged the same rate regardless of their purpose, the accounts they will be charged to or the person or office that placed them.

CHAPTER II

FULL-RANGE GAMES AND THEIR VALUES

2.1 Introduction

In this chapter we shall give most of the notations and definitions that will be used throughout this chapter and the following chapter, and we shall state and prove some results concerning full-range games.

Full-range games (to be defined shortly) arise naturally in the study of games generated by applications such as ours, and since the results are of interest by themselves, it was decided to put them here.

2.2 Preliminaries; Definitions of Games and Values

We will be dealing with non-atomic games almost exclusively and therefore we will give most of the definitions and notations here, with special definitions and notations appearing as they are needed. As the theory of non-atomic games was developed and written mostly by Aumann and Shapley, in particular in their book Values of Non-Atomic Games [1], we will be following the notations and definitions introduced there as much as possible.

Let $(I,\mathcal{C})$ be a measurable space; it will be fixed throughout. A set function is a real-valued function v on $\mathcal{C}$ satisfying $v(\emptyset) = 0$. The members of I are the players, the members of $\mathcal{C}$, coalitions, and a set function is a game. Following [1], we shall also assume the Standardness Assumption: $(I,\mathcal{C})$ is isomorphic to $([0,1], \mathcal{B})$ where $\mathcal{B}$ is the σ-algebra of the Borel sets of $[0,1]$.

A game v is monotonic if for all $S, T \in \mathcal{C}$, $S \subset T$ implies $v(S) \leq v(T)$, and it is of bounded variation if it is the difference between two monotonic games. The space of all games of bounded variation is

denoted BV. We define a norm on BV by

$$\|v\| = \inf(u(I) + w(I)) \tag{2.1}$$

where the inf ranges over all monotonic set functions u and w such that $v = u-w$. The quantity $\|v\|$ will be called the <u>variation</u> of v because of Proposition 2.4, and the norm defined in (2.1) will be referred to as the <u>variation norm</u>. Unless otherwise specified, all references to norms will be to this norm; other norms, when used, will utilize a different notation to distinguish them from the variation norm.

Before stating Proposition 2.4, we need the following

<u>Definition 2.2</u>: A non-decreasing sequence of sets of the form $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_m = I$ will be called a <u>chain</u>.

If v is a set function and Ω is a chain, then the variation of v over Ω is defined by

$$\|v\|_\Omega = \sum_{i=1}^{m} |v(S_i) - v(S_{i-1})|. \tag{2.3}$$

We can now state

<u>Proposition 2.4</u>: Let v be a set function. A necessary and sufficient condition that $v \in BV$ is that $\|v\|_\Omega$ be bounded over all chains Ω. If $v \in BV$ then

$$\|v\| = \sup \|v\|_\Omega, \tag{2.5}$$

where the sup is taken over all chains Ω.

Proof: This is Proposition 4.1 in [1].

Proposition 2.6: BV is complete, hence a Banach space.

Proof: This is Proposition 4.3 in [1].

Because of Proposition 2.4, we may, and indeed will, use the expressions given by (2.1) and (2.5) for $\|v\|$, without explicitly mentioning Proposition 2.4.

If Q is any space of games, Q^+ denotes the cone of the monotonic numbers of Q.

Definition 2.7: Let Q be a space of games. A mapping of Q into BV is called positive if it maps Q^+ into BV^+.

We now introduce a number of special subspaces of BV that will appear time and again in the sequel. FA will denote the subspace of BV consisting of all bounded, finitely additive set functions (i.e., the bounded, finitely additive signed measures on $(I,\mathcal{C})$). NA will denote the subspace of BV consisting of all the non-atomic measures on $(I,\mathcal{C})$. The space of non-atomic probability measures in $(I,\mathcal{C})$ will be denoted NA^1. The subspace of BV spanned by all powers of NA^+ measures will be denoted pNA; by this we mean that pNA is the closure of the set of all linear combinations of powers of NA^+ measures. DIAG will be defined as the set of all $v \in BV$ satisfying

(2.8) there is a positive integer k, a k-dimensional vector ζ of NA^1 measures and a neighborhood U in $\mathbb{R}^k$ of the diagonal $[0,\zeta(I)]$ such that if $\zeta(S) \in U$ then $v(S) = 0$.

Next we wish to introduce an important property that subspaces of

BV may have, namely symmetry. Let $\mathcal{G}$ be the group of automorphisms of the underlying space $(I,\mathcal{C})$, i.e., the one-to-one mappings of I onto itself that are measurable in both directions. Each $\theta \in \mathcal{G}$ induces a linear mapping θ_* of BV onto itself defined by $(\theta_* v)(S) = v(\theta S)$ for all $S \in \mathcal{C}$. A subspace Q of BV is called symmetric if $\theta_* Q = Q$ for all $\theta \in \mathcal{G}$.

Note: All the spaces defined so far are symmetric.

We now come to the definition of value.

Definition 2.9: Let Q be a symmetric subspace of BV. A value on Q is a positive linear operator ϕ from Q into FA satisfying the following two axioms:

$$(\phi v)(I) = v(I) \quad \forall\, v \in Q. \tag{2.10}$$

$$\phi\theta_* = \theta_*\phi \quad \forall\, \theta \in \mathcal{G}. \tag{2.11}$$

Axiom (2.10) is called the efficiency axiom (and will be referred to by this name often). It says that the value allocates to the players the entire amount available to the all-player set. Axiom (2.11) is called the symmetry axiom (and it, too, will often be called by that name). It says that the value does not depend on how the players are named.

The requirement that $\phi v \in$ FA is based partially on the desire that ϕv be a payoff vector, which is usually required to be, for any game, a member of FA.

At this point we need to introduce another space. Let Q be a symmetric subspace of BV, and let ϕ be a value on Q. We shall say that the pair (Q,ϕ) enjoys the diagonal property and the ϕ is a diagonal

value if $\phi v = 0$ for all $v \in Q \cap \mathrm{DIAG}$. We will define pNAD to be the closure of pNA + DIAG.

The last space to be introduced here will be pNA'. This space is defined as the closure, in the supremum norm, of the space of all linear combinations of polynomials in NA^+ measure; the supremum norm of a game v is defined by

$$\|v\|' = \sup\{|v(S)|: S \in \mathcal{C}\}.$$

The notation $\|\cdot\|'$ will be used, throughout, for the supremum norm, whenever it is needed.

The following two results will play an important role in this work.

Theorem 2.12: There is a unique value ϕ on pNA, and $\|\phi\| = 1$. Furthermore, let μ be a vector of measures in NA, and let f be continuously differentiable on R, the range of μ, with $f(0) = 0$. Then $f \circ \mu \in \mathrm{pNA}$ and, when R has full dimension,

$$\phi(f \circ \mu)(S) = \sum_{j=1}^{n} \mu_j(S) \int_0^1 f_j(t\mu(I))dt, \tag{2.13}$$

where $f_j = \partial f/\partial x_j$.

Proof. This is Theorem B in [1].

The notion of continuity of an operator, in the sequel, will be that of continuity in the variation norm.

Theorem 2.14: There is a unique continuous value on pNAD.

Proof: Proposition 43.13 in [1] asserts that there is a unique continuous diagonal value on pNAD. A recent result of A. Neyman [20] shows that continuous values are always diagonal.

Q.E.D.

The extension of these results is a major part of this work.

The games mentioned in Theorem 2.11, of the form $f \circ \mu$ where μ is a vector of NA measures and f is defined on the range of μ with $f(0) = 0$ are called vector-measure games. We will be concerned mainly with these games in the sequel.

Before we conclude this section, some notations and conventions--to be used throughout--should be mentioned. $\mathbb{R}^n$ will denote the n-dimensional Euclidean space; the jth unit vector of $\mathbb{R}^n$ (i.e., the vector whose jth coordinate is 1 and all the others are 0) will be denoted e^j; e will denote the vector in $\mathbb{R}^n$ of all ones. All vectors will be--where the distinction is important--column vectors, except for vectors of measures, which will be taken as row vectors. The inner product of two vectors x,y in $\mathbb{R}^n$ will be denoted $\langle x,y\rangle$. $\mathbb{R}^n_+$ will denote the non-negative orthant of $\mathbb{R}^n$. Finally, whenever the words game or games appear they will mean games on (I,C), unless otherwise specified and the notation Q will be reserved (again, unless otherwise specified) to mean a symmetric subspace of BV.

Last but not least in this list of preliminaries is

Theorem 2.15 (Laypunov's Theorem): The range of a non-atomic vector-measure is convex and compact.

Proof: See [18], or [6], or [15].

We will use this theorem often in the sequel, in fact it is of such fundamental importance to the whole theory of non-atomic games, that we may some times not even mention it when using it.

2.3 Values of Full-Range Games

In this section full-range games will be introduced and several results regarding values of such games will be presented.

<u>Definition 2.16</u>: Let μ be a vector of NA^1 measures, $\mu = (\mu_1,\dots,\mu_n)$. μ is said to have <u>full-range</u> if R, the range of μ, is the unit cube, i.e., $R \equiv \{\mu(S) | S \in C\} = [0,1]^n$.

<u>Definition 2.17</u>: Let $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfy $f(0) = 0$, and let $\mu = (\mu_1,\dots,\mu_n)$ have full-range. Then the game $f \circ \mu$ is called a <u>full-range</u> (vector-measure) <u>game</u>.

<u>Note</u>: Since every non-zero scalar NA^+ measure μ can be made full-range by taking $\mu' = \mu/\mu(I)$, every scalar measure game of a non-zero NA^+ measure is a full-range game. To see this, take $\mu' = \mu/\mu(I)$ and $f'(x) = f(\mu(I)\cdot x)$, $f' \circ \mu'$ is a full-range (scalar) measure game and $f' \circ \mu' = f \circ \mu$.

Let $\mu = (\mu_1,\dots,\mu_n)$ be a full-range measure. Since $R = [0,1]^n$, there exist $I_1,\dots,I_n \in C$ such that $\mu(I_j) = e^j$, $j = 1,\dots,n$. $I_1,\dots,I_n$ will be called <u>types</u>

We claim that the types are disjoint, except possibly for a set of (vector) measure 0. Indeed, we have for $i \neq j$, $\mu_k(I_i \cap I_j) \leq \min(\mu_k(I_i), \mu_k(I_j)) = 0$ for $k = 1,\dots,n$. Similarly, I differs from $\bigcup_{j=1}^{n} I_j$ at most by a set of (vector) measure 0. So, by subtracting from each I_j a finite number of sets of measure 0, we can get that $I_i \cap I_j = \emptyset$ for $i \neq j$, $i, j = 1,\dots,n$, and we will therefore assume that the types are indeed disjoint. We shall also assume, without loss of generality, (w.l.o.g.), that $I = \bigcup_{j=1}^{n} I_j$.

We can, therefore, represent each measurable set $T \in C$ by $(T_1,\dots,T_n)$ where $T_j = T \cap I_j$, $j = 1,\dots,n$, since $\bigcup_{j=1}^{n} T_j = \bigcup_{j=1}^{n} T \cap I_j = T \cap (\bigcup_{j=1}^{n} I_j) = T \cap I = T$. So we can write $T = (T_1,\dots,T_n)$, taking the equality sign symbolically.

Lemma 2.18: If $f \circ \mu$ is a full-range game and ϕ a value on $Q \ni f \circ \mu$, then for any $S, T \in \mathcal{C}$ such that $\mu(S) = \mu(T)$ we have $\phi(f \circ \mu)(S) = \phi(f \circ \mu)(T)$.

Proof: If $S, T \in \mathcal{C}$, $S \neq T$ but $\mu(S) = \mu(T)$, then we must have that $\mu_j(S) = \mu_j(T)$, $j = 1,\ldots,n$ and, since $\mu_j(S) = \mu_j(S_j)$ and $\mu_j(T) = \mu_j(T_j)$, we have that $\mu_j(S_j) = \mu_j(T_j)$, $j = 1,\ldots,n$.

There exist θ_j, a μ_j-measure-preserving automorphism on I_j, $j = 1,\ldots,n$ such that the symmetric difference $(\theta_j S_j \setminus T_j) \cup (T_j \setminus \theta_j S_j)$ is of μ_j-measure 0. This follows from [1], pp. 39, when μ_j is the Lebesque measure. When μ_j is not the Lebesque measure, the standardness assumption and Proposition 6.1 in [1] assure us that there is an automorphism Φ of $([0,1],\mathcal{B})$ such that $\Phi_* \mu_j = \lambda$, where λ is the Lebesque measure. So, if ψ is the Lebesque-measure preserving automorphism of I_j, then $\theta_j = \psi\Phi$.

Define $\theta\colon I \to I$ by $\theta x = \theta_j x \quad \forall\, x \in I_j$. Clearly, θ is a μ-measure-preserving automorphism on I, and therefore $\theta_* v = v$.

By our definition of θ we find that

$$\theta S = (\theta S \cap T) \cup (\theta S \setminus T), \quad \text{where} \quad \mu(\theta S \setminus T) = 0$$

and

$$T = (\theta S \cap T) \cup (T \setminus \theta S), \quad \text{where} \quad \mu(T \setminus \theta S) = 0,$$

so $\mu(\theta S) = \mu(T)$.

Since we know that null sets (here, sets with μ-measure 0) get value 0 (see [1], pp. 18), we conclude that

$$(\phi v)(T) = (\phi v)(\theta S) = \theta_*(\phi v)(S) = \phi(\theta_* v)(S) = (\phi v)(S).$$

Q.E.D.

This lemma says that, for full-range games, the value of a coalition is a function only of its measure. Before we can establish our next result about the values of full-range games, we need the following

Lemma 2.19: Let $f\colon \mathbb{R} \to \mathbb{R}$ be bounded on $[0,1]$ and suppose $g(x+y) = g(x) + g(y)$. Then $g(x) = xg(1)$.

Remark: This is a well-known result, but as we could find no reference to its proof, we provide one here.

Proof. The additivity of g implies the result immediately for $x = 0$, by $g(1) = g(1+0) = g(0) + g(1)$, implying $g(0) = 0 = 0g(1)$. It also is immediate for integers and rationals: for integers, by induction: $g(m) = g(m-1+1) = g(m-1) + g(1) = (m-1)g(1) + g(1) = mg(1)$; for rationals: $g(1) = g(1/2 + 1/2) = g(1/2) + g(1/2) = 2g(1/2)$, and similarly for all rationals. So, all we have to show is that for x irrational $0 < x < 1$, $g(x) = xg(1)$. W.l.o.g., assume $g(1) = 1$. Also assume that $g(x) = (1+c)x$, $c \neq 0$. Then for integer $m > 0$

$$g(m - \lfloor \tfrac{m}{x} \rfloor x) = m - \lfloor \tfrac{m}{x} \rfloor (1+c)x, \tag{2.20}$$

where $\lfloor y \rfloor$ = the greatest integer $\leq y$.

Rearranging (2.20) we get

$$g(m - \lfloor \tfrac{m}{x} \rfloor x) = (m - \lfloor \tfrac{m}{x} \rfloor x) - c\lfloor \tfrac{m}{x} \rfloor x. \tag{2.21}$$

Now, the expression $m - \lfloor \frac{m}{x} \rfloor x$ is a number between 0 and 1 for all positive integers m, while $\left| c\lfloor \frac{m}{x} \rfloor x \right| \to \infty$ as $m \to \infty$, since $c \neq 0$, thereby contradicting the boundedness of g on $[0,1]$.

Q.E.D.

Theorem 2.22: Let $v = f\circ\mu$ be a full-range game, and let ϕ be a value on $Q \ni v$. Then for $S \in \mathcal{C}$,

$$(2.23) \qquad (\phi v)(S) = \sum_{j=1}^{n} \mu_j(S)(\phi v)(I_j).$$

Proof: Lemma 2.18 assures us that if $S, T \in \mathcal{C}$ with $\mu(S) = \mu(T)$, the $(\phi v)(S) = (\phi v)(T)$. Let us then restrict our attention for a moment to the collection of measurable sets that are subsets of I_j, for some fixed j, that is to all $S \in \mathcal{C}$, $S \subset I_j$. We conclude that for such sets S, $(\phi v)(S) = g(\mu_j(S))$ for some function $g: \mathbb{R} \to \mathbb{R}$, additive and bounded on $[0,1]$. So, by Lemma 2.19 $g(x) = xg(1)$, or, in our case, $g(\mu_j(S)) = \mu_j(S)g(\mu_j(I_j))$. So, we conclude that $(\phi v)(S) = \mu_j(S)(\phi v)(I_j)$, and since for every set $S \in \mathcal{C}$, $S = (S_1,\ldots,S_n)$, where $\mu_j(S) = \mu_j(S_j)$, we conclude our result.

Q.E.D.

We thus see that for full-range games, once the value is determined on the types, it is completely determined. That is, if $T = (T_1,\ldots,T_n)$ and $\mu(T) = (\mu_1(T_1),\ldots,\mu_n(T_n)) \equiv (t_1,\ldots,t_n)$ then we have $\mu(T) = \sum_{j=1}^{n} t_j\mu(I_j)$ and so $\phi(f\circ\mu)(T) = \sum_{j=1}^{n} t_j\phi(f\circ\mu)(I_j)$. The function f contributes its share to the value determination by participating in the evaluation of the values for the types.

In the full-range case we have n disjoint types whose union is all of I and whose measures span $\mathbb{R}^n$. The next lemma--due, with its proof, to L. Shapley[1]--guarantees that we can always have the sets whose measures span $\mathbb{R}^n$ disjoint.

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[1] private communication.

Lemma 2.24: Let $T, R \in \mathcal{C}$ be such that $T \cap R \neq \emptyset$, and let μ be a vector of NA measures. Then there exist $R' \subset R$, $T' \subset T$ such that $R' \cap T' = \emptyset$ and $\mu(R') = 1/2\mu(R)$, $\mu(T') = 1/2\mu(T)$. (That is the sets R' and T' are disjoint and they have the same proportional measures as R and T.)

Proof. First let us note that in case we have inclusion, say $T \subset R$, this is an immediate result of Lyapunov's Theorem: taking $T' \subset T$ such that $\mu(T') = 1/2\mu(T)$ we obtain, since $\mu(R\backslash T') = \mu(R) - 1/2\,\mu(T) \geq 1/2\mu(R)$, that there exists $R' \subset R\backslash T$ with $\mu(R') = 1/2\mu(R)$.

If we do not have inclusion, let

$A_1 = T \cap R$

$A_2 = R\backslash T$

$A_3 = T\backslash R$.

Clearly $A_i \cap A_j = \emptyset$ for $i \neq j$.

By Lyapunov's Theorem we can find two disjoint subsets of A_i, say B_i and C_i such that $A_i = B_i \cup C_i$, $B_i \cap C_i = \emptyset$ and $\mu(B_i) = \mu(C_i) = 1/2\mu(A_i)$, $i = 1,2,3$. Now let $R' = B_1 \cup B_2$, $T' = C_1 \cup C_3$ and get that $R' \cap T' = \emptyset$ and that

$$\mu(R') = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) = \frac{1}{2}\mu(A_1) = \frac{1}{2}\mu(A_2) = \frac{1}{2}\mu(A_1 \cup A_2) = \frac{1}{2}\mu(R),$$

$$\mu(T') = \mu(C_1 \cup C_3) = \mu(C_1) + \mu(C_3) = \frac{1}{2}\mu(A_1) + \frac{1}{2}\mu(A_3) = \frac{1}{2}\mu(A_1 \cup A_3) = \frac{1}{2}\mu(T).$$

Q.E.D.

So far we have shown that when we have $\mu(S)$ expressed as a linear combination of $\mu(I_1),\ldots,\mu(I_n)$, then the value of S is the same linear combination, of values of $I_1,\ldots,I_n$. We would like to show that

when we select any n sets in C such that their measures span $\mathbb{R}^n$, we can still evaluate the value for those sets and then take the appropriate linear combinations of these values to compute the value of any coalition. The last lemma assures us that if we have sets $S_1,\ldots,S_n \in C$ such that $\mathrm{Span}\{\mu(S_1),\ldots,\mu(S_n)\} = \mathbb{R}^n$, then we can find $S_1' \subset S_1,\ldots,S_n' \subset S_n$ such that $S_i' \cap S_j' = \emptyset$ for $i \neq j$, and $\mathrm{Span}\{\mu(S_1'),\ldots,\mu(S_n')\} = \mathbb{R}^n$. So we can thus assume, w.l.o.g., that $S_i \cap S_j = \emptyset$ for $i \neq j$. Now if $S \in C$, then there exists a vector $\sigma = (\sigma_1,\ldots,\sigma_n)$ such that $\mu(S) = \sigma A$, where A is the matrix whose rows are $\mu(S_1),\ldots,\mu(S_n)$, that is

$$A = \begin{bmatrix} \mu(S_1) \\ \vdots \\ \mu(S_n) \end{bmatrix}$$

We also know that $(\phi v)(S_j) = \sum_{k=1}^{n} \mu_k(S_j)(\phi v)(I_k)$, or that

$$\begin{bmatrix} (\phi v)(I_1) \\ \vdots \\ (\phi v)(I_n) \end{bmatrix} = A^{-1} \begin{bmatrix} (\phi v)(S_1) \\ \vdots \\ (\phi v)(S_n) \end{bmatrix} . \tag{2.25}$$

So, since $\mu(S) = \sigma A$, $(\phi v)(S) = \sigma A \begin{bmatrix} (\phi v)(I_1) \\ \vdots \\ (\phi v)(I_n) \end{bmatrix}$ and, by (2.25) we find that

$$(\phi v)(S) = \sigma \begin{bmatrix} (\phi v)(S_1) \\ \vdots \\ (\phi v)(S_n) \end{bmatrix} . \tag{2.26}$$

So we have proved

<u>Theorem 2.27</u>: Let $S_1,\ldots,S_n \in C$ be disjoint sets whose measures

span $\mathbb{R}^n$. Let $S \in \mathcal{C}$ be such that $\mu(S) = \sigma A$, where σ and A are as above. Then (2.26) holds.

In summary, we have shown that full-range vector-measure games enjoy a very interesting property--their values are a function only of their measures, basically; we showed that for these games, any value is linear over the measures.

CHAPTER III

THE SPACE $\overline{GDD}$ AND ITS VALUE

3.1 Introduction

In this chapter a new space of games, $\overline{GDD}$, will be introduced and the existence of a unique continuous value on it will be proved. This space contains the "Rate-Setting Game" (to be introduced in the next chapter) and the value on the space--having the form given in (2.13)--will enable us to derive a (unique) solution to the Rate-Setting Problem of Section 1.2. The main result of this chapter is indeed the extension of Theorem 2.12 to this "larger" space.

Section 3.2 will prepare the ground for the main theorem in terms of definitions, notations and some preliminary results, while Section 3.3 will be devoted to the statement and proof of the main result as well as to the connection with Theorem 2.12.

3.2 The Space $\overline{GDD}$

Before we can introduce the space $\overline{GDD}$, we need a few preparatory definitions.

Definition 3.1: Let g be a real valued function on $\mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}$. The right (respectively, left) $j^{\underline{th}}$ partial derivative of g at $x \in \mathbb{R}^n$ is the limit

$$\lim_{t \to 0^+} \frac{g(x+te^j)-g(x)}{t} \quad (\text{resp. } \lim_{t \to 0^-} \frac{g(x+te^j)-g(x)}{t})$$

provided this limit exists and is finite.

Definition 3.2: DD_n, $n = 1,2,3,\ldots$, will denote the set of all functions $f: [0,1]^n \to \mathbb{R}$ having the following properties

(i) $f(0) = 0$.

(ii) f is continuous.

(iii) f is monotone non-decreasing on $[0,1]^n$.

(iv) There exists an integer $k \geq 0$ and points $\alpha_1,\ldots,\alpha_k \in [0,1]$ such that for each point on the diagonal $\{te\}_{0\leq t\leq 1}$ with the exception of $\alpha_1 e,\ldots,\alpha_k e$, there exists a neighborhood of the point where f is continuously differentiable; at the points $\alpha_1 e,\ldots,\alpha_k e$, the left and right partial derivatives of f exist; the left and right partial derivatives of f are uniformly bounded over the diagonal $\{te\}_{0\leq t\leq 1}$.

Let $DD = \bigcup_{n=1}^{\infty} DD_n$.

<u>Definition 3.3</u>: Let GDD be the subspace of BV generated by all games of the form $f\circ\mu$ where $\mu = (\mu_1,\ldots,\mu_n)$ with $\mu_j \in NA^1$ for $j = 1,\ldots,n$, and $f \in DD_n$. Let $\overline{GDD}$ be the closure--in the variation norm--of GDD.

We will be referring to the games that generate GDD as <u>generators</u>.

<u>Lemma 3.4</u>: GDD and $\overline{GDD}$ are symmetric.

<u>Proof</u>: Let $\theta \in \mathcal{G}$. It is enough to show that if $f\circ\mu$ is a generator of GDD, $\theta_*(f\circ\mu)$ is also a generator of GDD. Indeed, for any $S \in \mathcal{C}$, $\theta_*(f\circ\mu)(S) = (f\circ\mu)(\theta S)$, by definition of the operator θ_*. But, $(f\circ\mu)(\theta S) = (f\circ(\theta_*\mu))(S)$, and since $\mu = (\mu_1,\ldots,\mu_n)$ where $\mu_j \in NA^1$ for $j = 1,\ldots,n$, we have that $\theta_*\mu = (\theta_*\mu_1,\ldots,\theta_*\mu_n)$. Since we know that if $\mu_j \in NA^1$, we also have $\theta_*\mu_j \in NA^1$, we conclude that $f\circ(\theta_*\mu)$ is also a generator of GDD, as claimed.

Q.E.D.

<u>Definition 3.5</u>: A linear subspace Q of BV is called reproducing if $Q = Q^+ - Q^+$.

Lemma 3.6: GDD is reproducing.

Proof: Obvious since it is generated by monotonic games.

We will also need the following

Lemma 3.7: If $f \in DD$ then

$$\int_0^1 \left[\frac{d}{dt} f(te)\right]dt = f(e) - f(0) = f(e). \tag{3.8}$$

Proof: On the diagonal $\{te\}_{0 \leq t \leq 1}$ f is actually a function of one variable, t. Define $g(t) = f(te)$ for $0 \leq t \leq 1$. Then $g\colon [0,1] \to \mathbb{R}$, with $g(0) = 0$. So, if g is absolutely continuous, the lemma follows.

Let $\varepsilon > 0$ be given. We need to show that there exists a $\delta > 0$ such that if (a_i,b_i), $i = 1,\ldots,m$ are non-overlapping subintervals of $[0,1]$ with $\sum_{i=1}^{m} |b_i-a_i| < \delta$, then $\sum_{i=1}^{m} |g(b_i)-g(a_i)| < \varepsilon$.

Observe that $\frac{d}{dt}g(t) = \frac{d}{df} f(te) = \langle \nabla f(te), e \rangle = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(te)$.

Now, if ∇f exists at all points te with $t \in \bigcup_{i=1}^{m} (a_i,b_i)$ (that is none of the $\alpha_1,\ldots,\alpha_k$ appearing in the definition of DD_n belongs to $\bigcup_{i=1}^{m} (a_i,b_i)$) then we have

$$|g(b_i)-g(a_i)| = |g'(\zeta_i)||b_i-a_i|, \quad a_i \leq \zeta_i \leq b_i,$$

and since g' is then continuous on $\bigcup_{i=1}^{m} (a_i,b_i)$, we get that $|g'(\zeta_i)| \leq M \quad \forall_i$, for some $M > 0$. So

$$\sum_{i=1}^{m} |g(b_i)-g(a_i)| \leq M \sum_{i=1}^{m} |b_i-a_i| \leq M\delta,$$

and by taking $\delta = \frac{\varepsilon}{M}$ we are done.

Even if ∇f does not exist at some (finitely many) points of $\bigcup_{i=1}^{m} (a_i, b_i)$ (that is some-~~or~~-all of the points $\alpha_1, \ldots, \alpha_k$ belong to $\bigcup_{i=1}^{m} (a_i, b_i)$), we do know that the left and right partial derivatives exist and are uniformly bounded, say by M. So we conclude that $|g(b_i) - g(a_i)| \leq Mn|b_i - a_i|$, so here a $\delta = \frac{\varepsilon}{Mn}$ will do. So, by taking δ to be $\delta = \frac{\varepsilon}{Mn}$, the proof is completed.

Q.E.D.

We will also need, for the next section, the notion of extensions of set functions and their properties. Again, this will closely follow the notation and definitions of [1].

A measurable ideal set of $(I, \mathcal{C})$ (to be called an ideal set) is a measurable function from $(I, \mathcal{C})$ to $([0,1], \mathcal{B})$. Every "ordinary" measurable set $S \in \mathcal{C}$ has a natural ideal set corresponding to it, namely its characteristic function χ_S. The family of all measurable ideal subsets on $(I, \mathcal{C})$ will be denoted $\mathcal{I}$.

Define a partial order of $\mathcal{I}$ by $f \geq g$ if $f(s) \geq g(s)$ $\forall\, s \in I$. A real-valued function w on $\mathcal{I}$ with $w(0) = 0$ is called an ideal set function; it is called montonic if $f \geq g \Rightarrow w(f) \geq w(g)$.

We shall need the following

Proposition 3.9: There is a unique mapping that associates with each $v \in pNA'$ an ideal set function v^* such that:

(3.10) $(\alpha v + \beta w)^* = \alpha v^* + \beta w^*$ for $v, w \in pNA'$, $\alpha, \beta \in \mathbb{R}$;

(3.11) $(vw)^* = v^* w^*$ for $v, w \in pNA'$;

(3.12) $$\mu^*(f) = \int_I f d\mu \quad \text{for} \quad \mu \in NA;$$

(3.13) If f is a continuous real-valued function, then $(f o \mu)^* = f o \mu^*$.

Proof: (3.10), (3.11) and (3.12) are proved in Proposition 22.16 in [1], and (3.13) is also established there ((22.18)).

Q.E.D.

Define IBV to be the space of all ideal set functions of bounded variation, where an ideal set function is said to be of bounded variation if it is the difference of two monotonic ideal set functions. For $\tilde{v} \in$ IBV define

(3.14) $$\|\tilde{v}\| = \inf(u(\chi_I) + w(\chi_I)),$$

where the inf ranges over all montonic ideal set functions in IBV such that $\tilde{v} = u-w$. $\|\tilde{v}\|$ will be called the variation of $\tilde{v}$; it is easily seen to be a norm. This is completely analogous to the norm for BV, and indeed a similar result, justifying the term variation, holds:

(3.15) $$\|\tilde{v}\| = \sup \sum_{i=1}^{m} |\tilde{v}(g_i)-v(g_{i-1})|$$

where the sup ranges over all chains Ω of ideal sets $0 = g_0 \le g_1 \le \cdots \le g_m = \chi_I$.

We can now state

Proposition 3.16: If $v \in BV \cap pNA'$ then $\|v^*\| = \|v\|$.

Proof: This is Lemma 44.2 in [1].

Q.E.D.

We shall also need the following notation:

$$\partial v^*(t,S) = \frac{d}{d\tau} v^*(t\chi_I+\tau\chi_S)\Big|_{\tau=0}. \tag{3.17}$$

When $v = f\circ\mu$ and the partial derivatives of f exist and are continuous at $t\mu(I)$, $\partial v^*(t,S)$ then becomes, by Proposition 3.9 above

$$\partial v^*(t,S) = \sum_{j=1}^{n} \mu_j(S) \frac{\partial f}{\partial x_j}(t\mu(I)). \tag{3.18}$$

3.3 The Value on the Space $\overline{GDD}$

Our purpose in this section is to prove the following

Theorem 3.19: There is a unique continuous value ϕ on $\overline{GDD}$. Furthermore, if $v = f\circ\mu$ is a generator of GDD, then for all $S \in \mathcal{C}$

$$(\phi v)(S) = \sum_{j=1}^{n} \mu_j(S) \int_0^1 \frac{\partial f}{\partial x_j}(t\mu(I))dt = \left\langle \int_0^1 \nabla f(te)dt, \mu(S) \right\rangle, \tag{3.20}$$

where $\int_0^1 \nabla f(te)dt = \left(\int_0^1 \frac{\partial f}{\partial x_j}(te)dt, \ldots, \int_0^1 \frac{\partial f}{\partial x_n}(te)dt\right)$.

For this purpose, we will need a number of lemmas.

Lemma 3.21: ϕ, as given by Expression (3.20) is, indeed, a value on GDD.

Proof: It is enough to check the axioms for v a generator because, by definition, any $v \in$ GDD is a linear combination of generators. So, defining the value linearly--as we want it to be--will give us the desired outcome, providing we show that it is well defined.

We will thus proceed to show that the operator defined in (3.20)

satisfies the value axioms.

Efficiency:

$$(\phi v)(I) = \langle \int_0^1 \nabla f(te)dt, \mu(I) \rangle = \sum_{j=1}^{n} \mu_j(I) \int_0^1 \frac{\partial f}{\partial x_j}(te)dt$$

$$= \sum_{j=1}^{n} \int_0^1 \frac{\partial f}{\partial x_j}(te)dt = \int_0^1 \left[\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(te) \right] dt.$$

This last expression is equal, by the fact that f has a gradient on the diagonal except maybe for finitely many points, to $\int_0^1 \frac{d}{dt} f(te)dt$ and this, by Lemma 3.7 to $f(e)$, which in turn equals $f(\mu(I)) = v(I)$.

Symmetry:

$$\theta_*(\phi v)(S) = \theta_*(\phi(f o \mu))(S) = \varphi(f o \mu)(\theta S)$$

$$= \langle \int_0^1 \nabla f(te)dt, \mu(\theta S) \rangle = \phi(f o \theta_* \mu)(S) = \phi(\theta_* v)(S), \quad \forall\, \theta \in G.$$

Positivity:

Since $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of NA^1 measures, $f o \mu$ is monotone if and only if f is monotone non-decreasing which happens if and only if $\frac{\partial f}{\partial x_j} \geq 0 \;\; \forall\, j$. Thus

$$(\phi v)(S) = \langle \int_0^1 \nabla f(te)dt, \mu(S) \rangle = \sum_{j=1}^{n} \mu_j(S) \int_0^1 \frac{\partial f}{\partial x_j}(te)dt \geq 0.$$

To conclude the proof of the lemma, we have to show that indeed the operator we defined in (3.20) is well defined, i.e, it is independent

of the representation of v as $f \circ \mu$. For this we need

<u>Proposition 3.22</u>: For every monotonic game u and any positive, efficient operator ψ we have $\|\psi u\| = \|u\|$.

<u>Proof of Proposition 3.22</u>: By definition,

$$\|\psi u\| = \sup_{\Omega} \sum_{i=1}^{m} |(\psi u)(S_i) - (\psi u)(S_{i-1})|$$

where Ω is a chain $\emptyset = S_0 \subset S_1 \subset \ \ldots \subset S_m = I$.
Since ψ is positive, ψu is montonic, so

$$\|\psi u\| = (\psi u)(I) = u(I) = \|u\| \ ,$$

by the efficiency of ψ.

Q.E.D.

We now return to the proof of Lemma 3.21. Assume that $v = f \circ \mu$ and also $v = g \circ \nu$, where $\nu = (\nu_1, \ldots, \nu_m)$ is a vector of NA^1 measures and $g \in DD_m$.

Define $\zeta = (\mu, \nu)$ and $h(x,y) = f(x) - g(y)$. Then $h(\zeta(S)) = f(\mu(S)) - g(\nu(S)) = v(S) - v(S) = 0 \overset{def}{=} w(S)$. Clearly $h \circ \zeta \in GDD$, being the difference of two GDD games. Also, $w = h \circ \zeta$ is obviously monotonic so by Proposition 3.22, $\|\psi w\| = \|w\|$, since ϕ is positive and efficient.

Let $\alpha(S) = \langle \int_0^1 \nabla f(te)dt, \mu(S) \rangle$, $\beta(S) = \langle \int_0^1 \nabla g(te)dt, \nu(S) \rangle$ and $\gamma(S) = \langle \int_0^1 \nabla h(te)dt, \zeta(S) \rangle$. We see that $\gamma(S) = \alpha(S) - \beta(S) \quad \forall \ S \in \mathcal{C}$, i.e. that $\alpha - \beta = \gamma$. Consequently,

$$\|\alpha - \beta\| = \|\gamma\| = \|\phi w\| = \|w\| = \|0\| = 0,$$

so $\alpha = \beta$.

Q.E.D.

Next we wish to establish that the operator defined in (3.20) is the unique one having the properties of a value. We will be following the ideas of the proof of Theorem 6.4 in [21].

Lemma 3.23: The operator ϕ defined by (3.20) is the unique continuous value on GDD.

Proof: We have established already that (3.20) gives a value on GDD; now we wish to show it is the only continuous one. We again take v to be a generator of GDD, $v = f\circ\mu$.

Let $\alpha_1,\ldots,\alpha_k$ be the points of part (iv) of Definition 3.2 associated with f. (The points of discontinuity of the partial derivatives of f on the diagonal.) For simplicity, we shall establish the theorem first for $k = 1$ and then point out how the proof should be modified to handle any finite $k \geq 1$.

Let $\alpha = \alpha_1$ and let $0 < \delta < \frac{1}{2}\min(\alpha, 1-\alpha)$. Define the function F_δ: $[0,1] \to [0,1]$ as follows:

$$F_\delta(x) = \begin{cases} 0 & \text{if } |x-\alpha| \geq 2\delta \\ 1 & \text{if } |x-\alpha| \leq \delta \end{cases}$$

and for x such that $\delta < |x-\alpha| < 2\delta$, $F_\delta(x)$ is monotone increasing on $[\alpha-2\delta, \alpha-\delta]$ and monotone decreasing on $[\alpha+\delta, \alpha+2\delta]$, continuously differentiable, and $0 \leq F_\delta(x) \leq 1$.

Now define the game v_δ by

$$(3.24) \qquad v_\delta(S) = \prod_{j=1}^{n} (F_\delta \circ \mu_j)(S), \quad \forall\, S \in \mathcal{C}.$$

Since $F_\delta \circ \mu_j \in pNA \quad \forall j$ (see Theorem B of [1]) and since pNA is an algebra (see [1], pp. 50-54), $v_\delta \in pNA$. Note also that the definition of F_δ implies

$$(3.25) \qquad \|v_\delta\| \leq \prod_{j=1}^{n} \|F_\delta \circ \mu_j\| \leq \prod_{j=1}^{n} 2 = 2^n.$$

(For the first inequality, see [1], Proposition 4.3.)

Let U_δ be a subset of $\mathcal{C}$ defined as follows:

$$(3.26) \qquad U_\delta = \{S \in \mathcal{C}: |\mu_j(S)-\alpha| \leq 2\delta \quad \forall j\}.$$

Note that $v_\delta(S) = 0 \quad \forall\, S \notin U_\delta$.

Define the game $\tilde{v}_\delta$ by $\tilde{v}_\delta = v_\delta \cdot v$. We have that $\|\tilde{v}_\delta\| \leq \|v_\delta\| \cdot \|v\|$. In addition, we also have that

$$(3.27) \qquad \|\tilde{v}_\delta\| \leq \|v_\delta\| \cdot \|v\|_U,$$

where $\|v\|_U$ stands for the variation of v when the chains in the definition of the variation (2.3) are restricted to U, the subset of $\mathcal{C}$ where $v_\delta \neq 0$. We wish to show that $\|\tilde{v}_\delta\| \to 0$ as $\delta \to 0$. For this it will suffice to show that $\|v\|_{U_\delta} \to 0$ as $\delta \to 0$, because of (3.25) and (3.27).

Now, what is $\|v\|_{U_\delta}$? Let Ω be a chain that enters and leaves U_δ. Let i_0 be the first index for which $S_{i_0} \in U_\delta$, and let j_0 be the last index for which $S_{j_0} \in U_\delta$. Then we have that

$$(3.28)\qquad \|v\|_{\Omega} = \sum_{i=0}^{i_0-1} |v(S_i)-v(S_{i-1})| + \sum_{i=i_0}^{j_0} |v(S_i)-v(S_{i-1})|$$

$$+ \sum_{i=j_0+1}^{m} |v(S_i)-v(S_{i-1})|,$$

and the middle sum is the one that will give us $\|v\|_{U_\delta}$, when we take the supremum. Now, that sum reduces, by the monotonicity of v, to $v(S_{j_0}) - v(S_{i_0})$. Now, for any $S \in U_\delta$ we have $\alpha-2\delta \leq \mu_j(S) \leq \alpha+2\delta$. So, again by monotonicity of v, we conclude that

$$(3.29)\qquad \|v\|_{U_\delta} = \sup(v(S_{j_0}) - v(S_{i_0}))$$

$$\leq f((\alpha+2\delta)e) - f((\alpha-2\delta)e) \to 0 \text{ as } \delta \to 0.$$

Let $\mathcal{D}$ be the diagonal neighborhood defined by

$$(3.30)\qquad \mathcal{D} = \{S \in C\colon\ |\mu_j(S)-\mu_i(S)| < \delta \ \ \forall i,j\},$$

and let w_δ be the game

$$(3.31)\qquad w_\delta = v - \tilde{v}_\delta.$$

Let E be the union of all the neighborhoods whose existence is guaranteed in part (iv) of Definition (3.2)--where f is continuously differentiable and let $\overline{E} = \{S \in C | \mu(S) \in E\}$. Then, on $(\mathcal{D}\backslash U_\delta) \cap \overline{E}$, w_δ reduces to v which is continuously differentiable there. On $\mathcal{D} \cap U_\delta$

the construction of $\tilde{v}_\delta$ assures that w_δ is continuously differentiable there. So, we have found a neighborhood of the diagonal, namely $\{x: |x_i - x_j| < \delta \cap (E \cup \{x: |x_j - \alpha| < \delta\})$ where w_δ is continuously differentiable. So $w_\delta \in pNAD$. Therefore, by Theorem 2.14 and Proposition 44.22 ch. [1], there is a unique continuous value on pNAD, given by

$$(3.32)\quad \phi w_\delta(S) = \int_0^1 \partial w_\delta^*(t,S)dt = \int_0^{\alpha-2\delta} \partial v^*(t,S)dt + \int_{\alpha+2\delta}^1 \partial v^*(t,S)dt$$

$$+ \int_{\alpha-2\delta}^{\alpha+2\delta} \partial w_\delta^*(t,S)dt.$$

Now, as $\partial w_\delta^*(t,S) = \partial v^*(t,S) - \partial\tilde{v}_\delta^*(t,S)$, (3.32) becomes

$$(3.33)\quad \phi w_\delta(S) = \int_0^{\alpha-2\delta} \partial v^*(t,S)dt + \int_{\alpha+2\delta}^1 \partial v^*(t,S)dt + \int_{\alpha-2\delta}^{\alpha+2\delta} \partial v^*(t,S)dt$$

$$- \int_{\alpha-2\delta}^{\alpha+2\delta} \partial\tilde{v}_\delta^*(t,S)dt.$$

Over the intervals $[0, \alpha-2\delta]$ and $[\alpha+2\delta, 1]$ we have

$$\partial v^*(t,S) = \sum_{j=1}^n \mu_j(S) \frac{\partial f}{\partial x_j}(t\mu(I)),$$

so the continuity of the value will assure us that (3.33) will converge to (3.20) if $\int_{\alpha-2\delta}^{\alpha+2\delta} \partial\tilde{v}_\delta^*(t,S)dt \to 0$ as $\delta \to 0$. Now, as in (7.10) in [1], we can see that

$$\left| \int_{\alpha-2\delta}^{\alpha+2\delta} \partial\tilde{v}_\delta^*(t,S)dt \right| \leq \|\tilde{v}_\delta\|$$

which, as we saw above, tends to 0 as δ tends to 0.

So, the proof is complete for the case $k = 1$. The case $k > 1$ can now be easily handled by simply defining F_δ to have non-zero value around each of the k points of discontinuity of the partial derivatives along the diagonal.

Q.E.D.

So far we have exhibited the existence of a unique continuous value on GDD. To complete the proof of the theorem we now have to extend it--uniquely--to $\overline{\text{GDD}}$. For this purpose we need the following lemma which, with its proof, is due to A. Neyman.[1]

Lemma 3.34: Let $Q \subset BV$ be a symmetric subspace of games. If ϕ is a value on Q with $\|\phi\| \leq 1$, then there exists a unique continuous extension of ϕ to $\overline{Q}$.

Proof: First, let us show uniqueness. Let $v \in \overline{Q}$ and suppose $v = \lim_n v_n$ and also $v = \lim_n w_n$, v_n, $w_n \in Q$ $\forall n$. Define $\phi v = \lim_n \phi v_n$ (or $\phi v = \lim_n \phi w_n$). We have that $v = \lim_n v_n = \lim_n w_n$ or that $\lim_n (v_n - w_n) = 0$. Since ϕ is linear, $\phi(\lim_n (v_n - w_n)) = 0$ and since ϕ is continuous, $0 = \phi(\lim_n (v_n - w_n)) = \lim_n (\phi v_n - \phi w_n)$, so $\lim_n \phi v_n = \lim_n \phi w_n$.

Linearity of ϕ on $\overline{Q}$ is obvious from its definition as is the symmetry, so all that remains to be shown is positivity.

Assume ϕ is not positive. That is, assume there exists $v \in \overline{Q}$, v monotonic, such that ϕv is not monotonic. Indeed, let S, $T \in C$

[1] Private communication.

be such that $S \subset T$ but $(\phi v)(S) > (\phi v)(T)$. For the chain

$$\Omega: \emptyset = S_0 \subset S_1 = S \subset S_2 = T \subset S_3 = I$$

we have

$$\|\phi v\| \geq \sum_{i=1}^{3} |(\phi v)(S_i) - (\phi v)(S_{i-1})| > \sum_{i=1}^{3} [(\phi v)(S_i) - (\phi v)(S_{i-1})] = v(I) = \|v\|,$$

since $(\phi v)(S_i) - (\phi v)(S_i) < 0$. We thus get that $\|\phi v\| > \|v\|$ or that $\|\phi\| > 1$, a contradiction.

Q.E.D.

Finally, we need to show

<u>Lemma 3.35</u>: The value ϕ, on GDD, as defined by (3.20), has $\|\phi\| \leq 1$.

<u>Proof</u>: By Propositions 3.9 and 3.16 we know that v^* exists for all $v \in$ GDD and $\|v^*\| = \|v\|$. We therefore need to establish the following three facts, for $v \in$ GDD:

(i) $(\varphi v)(S) = \int_0^1 \partial v^*(t,S)dt \quad \forall\ S \in C$;

(ii) $\forall\ \varepsilon > 0 \quad \exists\ S \in C$ such that $|\phi v(S)| + |\phi v(I \backslash S)| \geq \|\phi v\| - \varepsilon$;

(iii) $\int_0^1 |\partial v^*(t,S)|dt + \int_0^1 |\partial v^*(t,I\backslash S)|dt \leq \|v^*\|$.

These three facts--once established--will mean that $\forall \varepsilon > 0 \quad \exists\ S \in C$ such that

$$(3.36) \quad \|\phi v\| - \varepsilon \leq |\phi v(S)| + |\phi v(I\backslash S)| = \left|\int_0^1 \partial v^*(t,S)dt\right| + \left|\int_0^1 \partial v^*(t,I\backslash S)dt\right| \leq \int_0^1 |\partial v^*(t,S)|dt + \int_0^1 |\partial v^*(t,I\backslash S)|dt \leq \|v^*\| = \|v\|.$$

:

So, we will get that $\forall\ \varepsilon > 0 \quad \|\phi v\| \leq \|v\| + \varepsilon$ or that $\|\phi v\| \leq \|v\| \quad \forall\ v \in GDD$, implying that $\|\phi\| \leq 1$, as desired.

Let us then establish those facts one by one:

(i) As before, we shall show this for v a generator, it will then follow for all $v \in GDD$ by the linearity properties of the extension and of the operator ∂.

Now, since the gradient of f exists on $\{te\}_{0\leq t\leq 1}$ with the possible exception of finitely many points, we have that

$$(3.37) \qquad \partial v^*(t,S) = f\mu(S)(te) = \sum_{j=1}^{n} \mu_j(S) \frac{\partial f}{\partial x_j}(te)$$

for almost all $t \in [0,1]$, since the range of μ is full-dimensional.

Therefore, (3.37) implies that

$$(3.38) \qquad (\phi v)(S) = \sum_{j=1}^{n} \mu_j(S) \int_0^1 \frac{\partial f}{\partial x_j}(te)dt = \int_0^1 [\sum_{j=1}^{n} \mu_j(S) \frac{\partial f}{\partial x_j}(te)]dt$$

$$= \int_0^1 \partial v^*(t,S)dt, \quad \forall\ S \in C,$$

as claimed.

(ii) Since we are dealing with a continuous operator, we have $\|\phi\| < \infty$ implying that $\|\phi v\| < K\|v\|$ for some $K > 0$. Now, from the definition of the variation norm (2.5) it follows that $\forall\ \varepsilon > 0$ there exists a chain $\Omega: \emptyset = S_0 \subset S_1 \subset \ldots \subset S_m = I$

such that $S_i \in \mathcal{C}\ \forall i$ and

(3.39) $$\|\phi v\| - \varepsilon \leq \sum_{i=1}^{m} |(\phi v)(S_i) - (\phi v)(S_{i-1})|.$$

Defining $T_i = S_i \backslash S_{i-1}$, $i = 1,\ldots,m$, (3.39) becomes

(3.40) $$\|\phi v\| - \varepsilon \leq \sum_{i=1}^{m} |(\phi v)(T_i)|.$$

Clearly, $T_i \in \mathcal{C}\ \ \forall i$.

Now, let $I^+ = \{i\colon (\phi v)(T_i) \geq 0\}$, $I^- = \{i\colon (\phi v)(T_i) < 0\}$, and define $S = \bigcup_{i\in I^+} T_i$. Then $I\backslash S = \bigcup_{i\in I^-} T_i$. We can now write (3.40) as

(3.41) $$\|\phi v\| - \varepsilon \leq |(\phi v)(S)| + |(\phi v)(I\backslash S)|,$$

as claimed.

(iii) Again, we will prove this fact for a generator. Since $v = f\circ\mu$ we know from Proposition 3.9 that $v^* = f\circ\mu^*$. We also know that $v \in BV \cap pNA'$--since a continuous function, f, can be approximated in the sup norm by polynomials ρ_n; then $\rho_n\circ\mu$ will approximate $f\circ\mu$ in the sup norm. Therefore, Lemma 44.11 together with 44.10 in [1] will give us that

(3.42) $$\int_0^1 |\partial v^*(t,S)|^+ dt + \int_0^1 |\partial v^*(t,I\backslash S)|^+ dt \leq \|v^*\|,$$

as needed, where $|\partial v^*(t,S)|^+ = \limsup_{\tau\to 0} \left|\dfrac{v^*(tX_I+\tau X_S)-v^*(tX_I)}{\tau}\right|,$

and obviously $|\partial v^*(t,S)| \leq |\partial v^*(t,S)|^+$, thus completing the proof.

Q.E.D.

Corollary 3.43: $pNA \subset \overline{GDD}$ and the value on $\overline{GDD}$ extends the value on pNA.

Proof: It is clear that powers of NA^+ measures are in GDD--by the definition of GDD. The rest is obvious.

Q.E.D.

CHAPTER IV

THE RATE-SETTING GAME AND ITS VALUE

4.1 The Rate-Setting Game

The problem was given to us in the following way. The monthly telephone usage data was given by the vector $\alpha = (\alpha_1,\ldots,\alpha_n)$ where

(4.1) α_j = the total number of minutes of calls of type j during the month, $j = 1,\ldots,n$,

where $n = 24 \times 2 \times k$, k being the number of different WATS bands, FX lines etc. in the system, 24 is for the number of hours in the day and the 2 comes from the fact that there are 2 types of billing days for the phone company--weekdays and weekends. That is, n is the number of different "types" of calls. (We had $n = 576$, as there were 12 different WATS band and FX lines.)

We now take I to be $I = [0,n)$, with the measurable sets $\mathcal{C}$ being the Borel sets. We take the subinterval $I_j = [j-1, j)$ to represent the calls of type j, $j = 1,\ldots,n$. We now define measures μ_j, $j = 1,\ldots,n$ on I by

$$(4.2) \qquad \mu_j(S) = \alpha_j \lambda(S \cap I_j),\ j = 1,\ldots,n,$$

where λ is the Lebesgue measure.

$\mu_j(S)$, for any $S \in \mathcal{C}$, is the total number of minutes of telephone calls of type j in S. For example, $\mu_{51}(S)$ may measure the total number of minutes of telephone calls in S that were placed during the month to WATS band 3 area, between 2 and 3 a.m. on a weekday. $\mu = (\mu_1,\ldots,\mu_n)$

is the vector of the measures. Note that μ is a vector of NA^+ measures.

Given a certain monthly load X on the system, $x = (x_1, \ldots, x_n)$, we can use the optimization routines developed in [9] to find the optimal--meaning least-cost--configuration of WATS and FX lines, etc., to service that load. We denote the cost of this configuration, including any "overhead", by $f(x_1, \ldots, x_n)$. We now have a game v on I defined by $v = f \circ \mu$; that is for each $S \in \mathcal{C}$, we have $v(S) = f(\mu(S))$, which is the minimal cost of servicing the demand for phone calls represented by S. Note that v is a vector-measure game.

It is clear that the situation is best described as a cooperative game. No single call merits the installation of a WATS line or an FX line, let alone the computerized device for accounting and security checks which costs a few thousand dollars a month just to lease and maintain. It is only the accumulation of a large number of calls--actually a large number of minutes of phone calls--that merits WATS or FX lines and the computerized device. Clearly, the system now affords savings to all calls, and the question here is--as it is most often in game theory--how to divide the "benefits of cooperation" among participating players. Various solution concepts exist which will usually give different answers to that basic question. The selection of one solution concept over the others depends usually on the conditions that the desired outcome--or allocation of benefits--must satisfy.

Let us, then, recall briefly the conditions that our solution must satisfy: it has to be efficient, fair, be in the form of a rate-structure and be unique.

Before showing--in the next section--how the value provides us with the desired solution, let us examine our game closely. We recall that for $i \neq j$, μ_i and μ_j are mutually singular and $\mu(I) = (\mu_1(I), \ldots, \mu_n(I)) = (\mu_1(I_1), \ldots, \mu_n(I_n))$, so by defining $\nu_j(S) = \frac{1}{\alpha_j} \mu_j(S)$ for $S \in \mathcal{C}$, we get

that the vector-measure $\nu = (\nu_1,\ldots,\nu_n)$ is a full-range measure. By defining the function

$$g(x_1,\ldots,x_n) = f(\alpha_1 x_1,\ldots,\alpha_n x_n)$$

we get that $v = g \circ \nu = f \circ \mu$, and $g \circ \nu$ is a full-range game.

We also observe that $v \in$ GDD. This follows from the fact that f is obviously continuous on the range of μ (and so, therefore, is g on the range of ν which is $[0,1]^n$); f is also--for a fixed configuration--continuously differentiable. So, if we had a single configuration serving every possible load in the range of μ, we would have $f \circ \mu \in$ pNA. However, this is clearly not the case, if only for the fact that for a very small load it is obviously optimal to have no system at all but rather to send all the calls DDD. Whenever we have a change of configurations, the left and right partial derivatives will exist, but they will not be the same. Note that they are always uniformly bounded--over the range of μ--by the DDD marginal costs.

Since we have only finitely many different configurations as we expand the system uniformly from 0 to $\mu(I)$ (that is along the diagonal $\{t\mu(I)\}_{0 \leq t \leq 1}$), we only have finitely many points of discontinuity of the partial derivatives of f on the diagonal, as required in the definition of GDD. We wish to comment also that, in the application, we considered it impossible to have two different optimal configurations along a segment of the diagonal. This, of course, would mean that on that segment of the diagonal discontinuities of the partial derivatives of f exist. The reason for that impossibility, or irrelevance to the actual application, is that a small perturbation--of seconds or even milliseconds in the total monthly load would

shift the diagonal and thus we would no longer have this problem. From a probabilistic point of view we can also say that the probability of a whole segment of the diagonal having discontinuities of the partial derivatives of f is 0, and therefore we can ignore it for all practical purposes.

4.2 The Value of the Rate-Setting Game

We now wish to show why the value of the Rate-Setting Game is indeed the right answer to our problem. First, we know that by Theorem 3.21 that there is a unique continuous value on GDD, and that if $v = f \circ \mu \in \mathrm{GDD}$ then the value is given by formula (3.22), so uniqueness is guaranteed.

Let us now examine formula (3.22) carefully. It says that

$$\phi(f \circ \mu)(S) = \sum_{j=1}^{n} \mu_j(S) \int_0^1 \frac{\partial f}{\partial x_j}(t\mu(I))dt.$$

We know that the μ_j's are mutually singular, meaning that

$$\mu_j(I_k) = 0 \quad \text{unless} \quad j = k.$$

That means that

$$\phi(f \circ \mu)(I_k) = \mu_k(I_k) \int_0^1 \frac{\partial f}{\partial x_k}(t\mu(I))dt. \tag{4.3}$$

Now, $\phi(f \circ \mu)(I_k)$ is interpreted here as the share of all calls of type k of the cost of operating the system. Noticing that (4.3) can be rewritten as

$$\phi(f \circ \mu)(I_k) = \alpha_k r_k, \tag{4.4}$$

where

(4.5) $$r_k = \int_0^1 \frac{\partial f}{\partial x_k}(t\mu(I))dt,$$

and α_k is the number of minutes of phone calls of type k in the month, we conclude that r_k is, indeed, the rate that must be charged to each minute of phone call of type k.

Note that efficiency and symmetry--or "fairness"--are also assured, by the value axioms. Also note that since our game is a full-range game, we indeed need calculate the value for each of the n different types, and then use linear combinations, as (3.22) does.

So, the r_k's, as given by (4.4), for $k = 1,\ldots,n$, are the correct rates that must be applied to the phone calls if the conditions imposed on the solution are to be met.

Another property of the rates given by (4.4) is that they are "time-of-day" prices. This is an added benefit of these rates; it follows from the fact that we have a rate, r_k, for each of our n different types. Since the types make time distinctions--meaning that two calls placed during different hours of the day are never in the same type--so will our rates. This also means that we can have the rate structure be as fine or as coarse as we wish by simply changing the data. For example, we could have rates that would change every half hour, or we could have rates that would change only three times a day--as the telephone company has today. It seems, though, that hourly rates are most appropriate and that, used properly, they can save both the system and the individual phone user some money in addition to the savings already realized by using the system and the WATS and FX lines.

Computational procedures, as well as a numerical example are presented elsewhere [2]. It might be of interest, though, to mention that the recommendations generated by this study and actual rates produced by it (similar to the ones given in [2]), were actually implemented for the billing of calls going through the system.

CHAPTER V

VALUES AND CORES OF NON-ATOMIC LINEAR PRODUCTION GAMES

5.1 Introduction

Linear production games were introduced by Owen in [22] for a finite number of players. The setting of Owen's work is as follows: Let $c = (c_1, \ldots, c_p)$ be a given set of fixed prices for p different products; let $b^j = (b^j_1, \ldots, b^j_m)$ be the initial allocation, to player j, of raw materials, $1, \ldots, m$ for $j \in N = \{1, \ldots, n\}$; let A be a production matrix, $A = (a_{ki})$, $k = 1, \ldots, p$, $i = 1, \ldots, m$ where a_{ki} units of raw material i are needed in the production of one unit of final product k. Using the notation $b(S) = \sum_{j \in S} b^j = (b_1(S), \ldots, b_m(S)) = (\sum_{j \in S} b^j_1, \ldots, \sum_{j \in S} b^j_m)$, we can now define a linear production game, as follows: for all $S \in 2^N - \emptyset$

$$
\begin{aligned}
& v(S) = \max \langle c, x \rangle \\
(5.1) \qquad & \text{s.t. } xA \le b(S) \\
& x \ge 0,
\end{aligned}
$$

where $x = (x_1, \ldots, x_p)$. The interpretation is that each coalition S tries to find the production schedule x that is feasible to it $(Ax \le b(S))$ and that will bring the maximal revenue $\langle c, x \rangle$. We assume that there is no demand for the raw materials as such, but only for the final goods, and that the prices of those are fixed and cannot be changed by the players. Using the duality theory of linear programming, Owen establishes in [22] the following results, which we summarize briefly here.

Let us consider the dual problem of (5.1) when we take the coalition S to be $S = N$, the grand coalition. The dual is

$$\min \langle y, b(N) \rangle$$

$$\text{(5.2)} \qquad \text{s.t. } Ay \geq c$$

$$y \geq 0$$

Observing that the dual constraints are independent of the coalition, making them identical for all $S \in 2^N - \emptyset$, Owen concludes that if y^* is an optimal solution to (5.2) then the following is true:

a. $v(N) = \langle y^*, b(N) \rangle$

b. $v(S) \leq \langle y^*, b(S) \rangle \quad \forall \, S \in 2^N - \emptyset$.

Hence, the imputation (5.3) $u = (u_1, \ldots, u_n) = (\langle y^*, b^n \rangle, \ldots, \langle y^*, b \rangle)$ is in the core of v.

So, we see that if y^* is any optimal solution to (5.2), then the imputation defined by (5.3) belongs to the core of v. However, the core may consist of many other points that are not of the form (5.3). Since a y^* that (optimally) solves (5.2) is called a vector of shadow prices (equilibrium prices by Owen in [22]), the fact that there are imputations in the core that are not of the form (5.3) means that these imputations are not given by a vector of shadow prices. This problem, however, disappears when the game is replicated an infinite number of times or, if y^* that solves (5.2) is unique, when the game is replicated a large--but finite--number of times.

An interesting problem is that of trying to generalize these results to the non-atomic case. This is done in the next Section, 5.2. In

addition to generalizing the results about the core of the finite case to the core of the non-atomic case, a relationship between the core and the value of the non-atomic game is derived for a special case.

In Section 5.3, a brief summary is presented, together with a comparison between the results for the finite linear production games and for the non-atomic linear production game. The differences are also exhibited by using a few examples.

5.2 Non-Atomic Linear Production Games

Let $\mu = (\mu_1,\ldots,\mu_n)$ be a vector of NA^+ measures on the underlying space $(I,\mathcal{C})$. Let A be an $m \times n$ matrix (a_{ij}) with $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} > 0$. Let $c = (c_1,\ldots,c_m)$ be a positive vector, i.e., $c_i > 0$ for $i = 1,\ldots,m$.

Define a function $f\colon \mathbb{R}_+^n \to \mathbb{R}$ as follows:

$$f(x) = \max\langle c,z\rangle$$

$$\text{s.t. } zA \leq x \tag{5.4}$$

$$z \geq 0.$$

We need to show that f is well-defined. Note that since $c_i > 0 \ \forall_i$ and $z \geq 0$, $\langle c,z\rangle \geq 0$ and, since $x \geq 0$, and $a_{ij} \geq 0 \quad \forall\, i,j$, $z = 0$ is always feasible. So, if we let f take on values in the extended real line, we see that it is indeed well defined.

Let us now look at the dual program of (5.4)

$$\min\langle y,x\rangle$$

(5.5)
$$\text{s.t. } Ay \geq c$$

$$y \geq 0.$$

Since $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} > 0$ this program is clearly feasible, as can be seen by taking y large enough. So by the duality theorem of linear programming we know that both programs (5.4) and (5.5) have finite optimal solutions and that there exist y^* feasible for (5.5) and z^* feasible for (5.4) such that

(5.6)
$$f(x) = \langle c,z^*\rangle = \langle y^*,x\rangle.$$

So, f is well-defined and it is actually given by

(5.7)
$$f(x) = \langle y^*,x\rangle$$

where y^* is an optimal solution to (5.5).

Definition 5.8: A real-valued function g on $\mathbb{R}^n$ is called superadditive if for all x and y in $\mathbb{R}^n$

$$g(x+y) \geq g(x) + g(y).$$

Definition 5.9: A real valued function g is called positively homogeneous of degree 1 if for all x in the domain of g and $\alpha \in \mathbb{R}_+$

$$g(\alpha x) = \alpha g(x).$$

Lemma 5.10: The function f defined in (5.4) is superadditive and positively homogeneous of degree 1.

Proof: We need to show that for $x,y \in \mathbb{R}^n_+$, $f(x+y) \geq f(x) + f(y)$.

$$\begin{aligned} f(x+y) = \max \langle c,z \rangle \\ \text{s.t. } zA \leq (x+y) \\ z \geq 0. \end{aligned} \tag{5.11}$$

$$\begin{aligned} f(x) = \max \langle c,u \rangle \\ \text{s.t. } uA \leq x \\ u \geq 0. \end{aligned} \tag{5.12}$$

$$\begin{aligned} f(y) = \max \langle c,w \rangle \\ \text{s.t. } wA \leq y \\ w \geq 0. \end{aligned} \tag{5.13}$$

Let u^* and w^* be optimal solutions to (5.12) and (5.13) respectively. This means in particular that u^* and w^* are feasible for (5.12) and (5.13) respectively, meaning that

$$u^*A \leq x \quad \text{and} \quad w^*A \leq y,$$

and that $u^*,w^* \geq 0$. So

$$(u^*+w^*)A = u^*A + w^*A \leq x + y,$$

and $u^* + w^* \geq 0$. So $u^* + w^*$ is feasible for (5.11). Now, since $f(x+y)$ is the maximum of the inner product of c with any feasible vector for (5.11) we get in particular that

$$f(x+y) \geq \langle c,u^*+w^*\rangle = \langle c,u^*\rangle + \langle c,w^*\rangle = f(x) + f(y),$$

proving that f is indeed superadditive.

To show that f is positively homogeneous of degree 1, let $t > 0$ be given. (For $t = 0$ nothing need be proved, since $f(0) = 0$.)

$$\begin{aligned} f(tx) = \max &\langle c,z\rangle \\ \text{s.t. } &zA \leq tx \\ &z \geq 0. \end{aligned} \tag{5.14}$$

Let y^* be an optimal solution to the dual of (5.12), the program for $f(x)$. This dual is

$$\begin{aligned} \min &\langle y,x\rangle \\ \text{s.t. } &Ay \geq c \\ &y \geq 0. \end{aligned} \tag{5.15}$$

By (5.7) we know that $f(x) = \langle y^*,x\rangle$ where y^* satisfies the dual constraints of (5.15), i.e., $Ay^* \geq c$, $y^* \geq 0$. Since these constraints

are independent of x, y^* is also feasible for the dual of (5.14), the program for $f(tx)$. Thus $f(tx) \le \langle y^*, tx\rangle = t\langle y^*,x\rangle = tf(x)$. Similarly, $f(x) = f(t^{-1}(tx)) \le t^{-1}f(tx)$, so $f(tx) = tf(x)$ and f is indeed positively homogeneous of degree 1.

Q.E.D.

Lemma 5.16: Let $x \in \mathbb{R}^n$, $x > 0$, and suppose y^* is the unique optimal solution to (5.15), the dual of (5.12) (the program for $f(x)$). Then f is continuously differentiable at x and $\frac{\partial f}{\partial x_j}(x) = y_j^*$.

Proof: Let $y^*,y^1,\ldots,y^E$ be the extreme points of $\{Ay \ge c,\ y \ge 0\}$. Since y^* is, by assumption, the unique solution to

$$\begin{aligned} &\min \langle y,x\rangle \\ (5.17)\qquad &\text{s.t. } Ay \ge c \\ &\qquad y \ge 0, \end{aligned}$$

we know that $\langle y^*,x\rangle < \langle y^\ell,x\rangle$, $\ell = 1,\ldots,E$.

Let $\varepsilon = \min\limits_{\ell=1,\ldots,E} (\langle y^\ell,x\rangle - \langle y^*,x\rangle)$; $\varepsilon > 0$ by assumption. Now, for $j = 1,\ldots,m$, define δ_j^1,

$$(5.18)\qquad \delta_j^1 = \min\left\{\left|\frac{\varepsilon}{y_j^*-y_j^\ell}\right| : y_j^* - y_j^\ell \ne 0,\ \ell = 1,\ldots,E\right\},$$

and define δ_j by

$$(5.19)\qquad \delta_j = \min(\delta_j^1, x_j).$$

Then for $0 < |t| < \delta_j$, we have by (5.19) $x + te^j > 0$, and by (5.18),

$$\langle y^*-y^\ell,\ x+te^j\rangle = \langle y^*-y^\ell,\ x\rangle + t(y_j^*-y_j^\ell) < 0, \quad \text{for } \ell = 1,\ldots,E.$$

So, y^* is the unique optimal solution for the dual of the program for $f(x+te^j)$. Hence

$$\text{(5.20)} \qquad \frac{f(x+te^j)-f(x)}{t} = \frac{\langle y^*, x+te^j\rangle - \langle y^*, x\rangle}{t} = \frac{ty_j^*}{t} = y_j^*.$$

Now we are done, since $\frac{\partial f}{\partial x_j}(x) = \lim_{t\to 0} \frac{f(x+te^j)-f(x)}{t} = y_j^*$, as claimed.

Q.E.D.

We are now in a position to define the non-atomic linear production game v on $(I,\mathcal{C})$, by $v = f\circ\mu$, or $v(S) = f(\mu(S))$ $\forall\, S \in \mathcal{C}$. Explicitly,

$$\text{(5.21)} \qquad \begin{aligned} v(S) = f(\mu(S)) = \max&\langle c,x\rangle \\ \text{s.t. } xA &\leq \mu(S) \\ x &\geq 0. \end{aligned}$$

Clearly, v so defined depends on μ,A and c, but once these are specified, we can talk about "the" non-atomic linear production game.

Before stating our next result, we need the following

<u>Definition 5.22</u>: Let v be a game. The <u>core</u> of v consists of all members $\lambda \in FA$ such that $\lambda(S) \geq v(S)$ $\forall\, S \in \mathcal{C}$ and $\lambda(I) = v(I)$.

We know that if y^* is an optimal solution to the dual of (5.21), then $v(S) = f(\mu(S)) = \langle y^*,\mu(S)\rangle$. This leads us to

<u>Theorem 5.23</u>: Let $\hat{y}$ be an optimal solution to the dual of (5.21) for $S = I$. (I.e., $v(I) = \langle\hat{y},\mu(I)\rangle$, $Ay \geq c$, $\hat{y} \geq 0$.) Then

$\nu_{\hat{y}}(\cdot) = \langle \hat{y}, \mu(\cdot) \rangle$ is a non-atomic measure on $(I,\mathcal{C})$ that belongs to the core of v, that is, $\nu_{\hat{y}}(S) \geq v(S) \quad \forall\, S \in \mathcal{C}$, $\nu_{\hat{y}}(I) = v(I)$.

Proof: The constraint set for the dual of (5.21) is the same for all $S \in \mathcal{C}$; it is $\{Ay \geq c,\ y \geq 0\}$. Therefore, since $\hat{y}$ is optimal for the dual program of (5.21) for $S = I$, it is necessarily feasible, and thus it is feasible for the dual of (5.21) for all $S \in \mathcal{C}$. Now, for all $S \in \mathcal{C}$, $v(S)$ is given by (5.21) and by the duality theorem of linear programming $v(S)$ also equals

$$\begin{aligned} &\min \langle y, \mu(S) \rangle \\ (5.24) \qquad &\text{s.t. } Ay \geq c \\ &\qquad y \geq 0, \end{aligned}$$

being the dual of (5.21). Hence since $\hat{y}$ is feasible for (5.24) we have in particular that

$$(5.25) \qquad v(S) \leq \langle \hat{y}, \mu(S) \rangle = \nu_{\hat{y}}(S).$$

It is clear that $\nu_{\hat{y}}$ is a non-atomic measure on $(I,\mathcal{C})$ and (5.24) shows that $\nu_{\hat{y}}(S) \geq v(S) \quad \forall\, S \in \mathcal{C}$; since we also have $\nu_{\hat{y}}(I) = v(I)$, we see that indeed $\nu_{\hat{y}}$ belongs to the core of v.

Q.E.D.

Note: Since optimal solutions to the dual always exist, we have shown that the core of the linear production game is never empty. Moreover, this holds for every subgame as well, meaning that every subgame has a non-empty core.

So far, the theorem and the proof are very similar to Owen's, because we have not utilized the non-atomic nature of the initial allocation measure μ. However, the non-atomicity of μ plays a crucial role in the next theorem which gives us the value of linear production games in a special case.

Theorem 5.26: If y^* is the unique optimal solution for (5.24) for $S = I$ (i.e., the unique optimal dual solution for $S = I$), then ν_{y^*} is also the value of the game v.

Proof: Lemma 5.10 shows that f is superadditive and positively homogeneous of degree 1. Therefore, since y^* is the unique optimal dual solution for $v(I) = f(\mu(I))$, it is also the unique optimal dual solution for $f(t\mu(I))$, $0 < t \leq 1$. Now, since for $0 < t \leq 1$, $t\mu(I) \geq 0$, there exist an open ball around $t\mu(I)$, with radius $\delta > 0$ given by

$$(5.27) \qquad \delta = \min_{j=1,\ldots,n} \delta_j ,$$

where the δ_j's are given by (5.19). Therefore, by taking the union of all those open balls around $t\mu(I)$, $0 < t < 1$, we get an open neighborhood U around $\{t\mu(I): 0 < t \leq 1\}$, on which f is continuously differentiable--by Lemma 5.16, and for $0 < t \leq 1$, $\frac{\partial f}{\partial x_j}(t\mu(I)) = y_j^*$, $j = 1,\ldots,n$.

Now, since $v = f\circ\mu$ is continuously differentiable on a neighborhood of the diagonal, it belongs to pNAD. (To see this, just observe that we could redefine v to be $f\circ\mu$ on U, and extend f to be continuously differentiable on all of R, the range of μ, thus making it a member of pNA.)

We also claim that $v = f\circ\mu \in$ pNA'. This follows from the fact that

f is continuous--being a piecewise linear function--and so can be approximated in the sup norm by polynomials, i.e., there exist polynomials $P_1, P_2, \ldots,$ such that $\|P_n - f\|' \to 0$ as $n \to \infty$; this implies that $\|P_n \circ \mu - f \circ \mu\|' \to 0$ as $n \to \infty$, proving that $f \circ \mu \in pNA'$.

So, $v \in pNAD \cap pNA'$, it is positively homogeneous of degree 1 and is superadditive. Hence, by Proposition 44.28 in [1], there is a unique member in the core of v, namely the value ϕv. Q.E.D.

Remark: Since we now have that ν_y* is the value, it may seem, at first look, that we have two conflicting formulas for the value of v. One is

$$(\phi v)(S) = \sum_{j=1}^{n} \mu_j(S) \int_0^1 \frac{\partial f}{\partial x_j} (t\mu(I))dt,$$

the other

$$(\phi v)(S) = \nu_y*(S) = \langle y^*, \mu(S) \rangle .$$

However, since $\frac{\partial f}{\partial x_j} (t\mu(I))dt = y_j^*$, for $0 < t \leq 1$, the formulas coincide.

We have thus shown that when the linear production game is non-atomic, the uniqueness of the optimal dual solution for the program for the whole player set not only guarantees that ν_y* will be the unique member of the core but also that it will be the value of the game. In the next section we will elaborate further on this, and give examples that point out the differences between finite linear production games and non-atomic linear production games.

We now approach the problem of fully characterizing the cores of non-atomic linear production games. At this stage we know that if $y^1, \ldots, y^L$

are the extreme points of $\{Ay \geq c,\ y \geq 0\}$ at which $v(I)$ is attained and if $y \in \mathrm{conv}\{y^1,\ldots,y^L\}$ then the measure ν_y defined by $\nu_y(S) = \langle y,\mu(S)\rangle$ belongs to the core of v. The question now is whether there are other measures in the core of v. We shall show that for non-atomic linear production game the answer is no. For this purpose, we will need the following three more general results about measures in the cores of non-atomic vector-measure games.

Lemma 5.28: Suppose $v = f \circ \mu$ is a non-atomic game which is positively homogeneous of degree 1. Then every measure λ in the core of v is a function of the measure μ, namely if $S, T \in \mathcal{C}$ are such that $\mu(S) = \mu(T)$, then $\lambda(S) = \lambda(T)$.

Proof: Let us first establish the fact that λ is homogeneous on the diagonal $[0,\mu(I)]$, that is, if $S \in \mathcal{C}$ is such that $\mu(S) = s\mu(I)$, then $\lambda(S) = s\lambda(I) = sv(I)$. For this purpose, let $S \in \mathcal{C}$ be such that $\mu(S) = s\mu(I)$, $0 \leq s \leq 1$. Since if $s = 0$ or $s = 1$ the fact is obvious, assume that $0 < s < 1$. Since λ is in the core of v, we must have

$$(5.29)\quad \lambda(S) \geq v(S) = f(\mu(S)) = f(s\mu(I)) = sf(\mu(I)) = sv(I).$$

Similarly we must have

$$(5.30)\quad \lambda(I\backslash S) \geq v(I\backslash S) = f(\mu(I\backslash S)) = f((1-s)\mu(I)) = (1-s)f(\mu(I))$$

$$= (1-s)v(I).$$

Adding the inequalities (5.29) and (5.30) we get

$$(5.31)\qquad \lambda(I) = \lambda(S) + \lambda(I/S) \geq sv(I) + (1-s)v(I) = v(I).$$

But, we know that $\lambda(I) = v(I)$. Therefore, we must have equalities in (5.24) and (5.25), establishing our fact.

Let us now proceed with the proof of the lemma. Let $S, T \in \mathcal{C}$ be such that $\mu(S) = \mu(T)$. W.l.o.g. we can assume that $S \cap T = \emptyset$. If not, then since $S = (S\cap T) \cup (S\setminus T)$ and $T = (T\cap S) \cup (T\setminus S)$, and $\mu(S\setminus T) = \mu(T\setminus S)$, the problem would reduce to showing that $\lambda(T\setminus S) = \lambda(S\setminus T)$. We can also assume that $\mu(S) \neq s\mu(I)$ for some $0 \leq s \leq 1$, for otherwise the result would follow from the first part of the proof.

Let $W = I\setminus(S\cup T)$, and let $V \subset W$ be such that $\mu(V) = \frac{1}{2}\mu(W)$. Clearly $S \cap V = \emptyset = T \cap V$. Define now $S' = S \cup V$, $T' = T \cup V$.

$$\mu(S') = \mu(S) + \mu(V) = \mu(S) + \frac{1}{2}\mu(I\setminus(S\cup T)) =$$

$$= \mu(S) + \frac{1}{2}[\mu(I) - 2\mu(S)] = \frac{1}{2}\mu(I),$$

and clearly $\mu(T') = \mu(S')$. Now, by the first part of the proof, we know that $\lambda(S') = \frac{1}{2}\lambda(I) = \lambda(T')$. On the other hand, $\lambda(S') = \lambda(S) + \lambda(V)$ and $\lambda(T') = \lambda(T) + \lambda(V)$. So, we conclude that

$$\lambda(T) + \lambda(V) = \lambda(S) + \lambda(V)$$

or that $\lambda(S) = \lambda(T)$, as claimed.

Q.E.D.

<u>Lemma 5.32</u>: Let $f \circ \mu$ be positively homogeneous of degree 1, with f continuous on R, the range of μ, and suppose λ belongs to the

core of $f\circ\mu$. Then if $S, S_1, \ldots, S_n \in \mathcal{C}$ and $\mu(S) = \sum_{j=1}^{n} \alpha_j \mu(S_j)$, then $\lambda(S) = \sum_{j=1}^{n} \alpha_j \lambda(S_j)$.

<u>Proof</u>: Because λ belongs to the core of $f\circ\mu$ it is finitely additive, and so the proof reduces to showing that if $\mu(S) = \alpha\mu(S_1)$, then $\lambda(S) = \alpha\lambda(S_1)$ $\forall\, \alpha \in \mathbb{R}$.

By the preceding lemma and Lyapunov's Theorem, this holds for all rationals between 0 and 1. Since f is continuous then for any monotone increasing sequence $\{T_i\}_{i=1}^{\infty} \in \mathcal{C}$ whose union is I we have $(f\circ\mu)(T_i) \to (f\circ\mu)(I)$ as $i \to \infty$. Hence by Lemma A in [23], (or Theorem 3.2 in (24]) every outcome in the core of $f\circ\mu$ is σ-additive. Now let α be an irrational number between 0 and 1. Let $0 \leq r_1 \leq r_2 \leq \ldots \leq r_i \leq \ldots \leq \alpha$ be a montone increasing sequence of rationals in $[0,\alpha]$, whose limit is α. Let $T_i \in \mathcal{C}$ be such that $T_i \subset T_{i+1}$, $\forall_i$ and such that $\mu(T_i) = r_i\mu(S_1)$. Clearly $\mu(\bigcup_i T_i) = \alpha\mu(S_1)$. If we now let $W_i = T_i \setminus T_{i-1}$, $\forall_i$, (with $T_0 = \emptyset$), we also get that $\mu(\bigcup_i W_i) = \alpha\mu(S_1)$. Now, since λ is σ-additive we obtain that

$$\lambda(S) = \sum_{i=1}^{\infty} \lambda(W_i) = \sum_{i=1}^{\infty} \lambda(T_i \setminus T_{i-1}) = \lim_{i\to\infty} \lambda(T_i)$$

$$= \lim_{i\to\infty} r_i\, \lambda(S_1) = \lambda(S_1) \lim_{i\to\infty} r_i = \alpha\lambda(S_1),$$

as claimed.

Once we have the lemma proved for every $0 \leq \alpha \leq 1$, we also have it for $\alpha > 1$, for if $\mu(S) = \alpha\mu(S_1)$ implies $\lambda(S) = \alpha\lambda(S_1)$ for $0 \leq \alpha \leq 1$ then, for $\alpha > 1$ we have that $\frac{1}{\alpha}\mu(S) = \mu(S_1)$, with $\frac{1}{\alpha} < 1$, so now we have $\frac{1}{\alpha}\lambda(S) = \lambda(S_1)$ or $\lambda(S) = \alpha\lambda(S_1)$. So, it remains to

establish the lemma for $\alpha < 0$. In this case we have

$$\mu(S) = -|\alpha|\mu(S_1).$$

But this may be rewritten as

$$\mu(S) + |\alpha|\mu(S_1) = 0 = \mu(\emptyset).$$

So

$$\lambda(S) + |\alpha|\lambda(S_1) = \lambda(\emptyset) = 0$$

or

$$\lambda(S) = -|\alpha|\lambda(S_1) = \alpha\lambda(S_1),$$

as we needed to show.

Q.E.D.

Lemma 5.33: Let $f \circ \mu$ be positively homogeneous of degree 1 and suppose λ belongs to the core of $f \circ \mu$. Then there exists $d \in \mathbb{R}^n$ such that $\forall\, S \in \mathcal{C}$, $\lambda(S) = \langle \mu(S), d \rangle$.

Proof: Let $S_1, \ldots, S_k \in \mathcal{C}$ be such that $\mu(S_1), \ldots, \mu(S_k)$ span the k-dimensional subspace generated by the range of μ.
Let M be the matrix

$$M = \begin{bmatrix} \mu(S_1) \\ \vdots \\ \mu(S_k) \end{bmatrix} = \begin{bmatrix} \mu_1(S_1) & \cdots & \mu_n(S_1) \\ \vdots & & \\ \mu_1(S_k) & & \mu_n(S_K) \end{bmatrix},$$

and let D be the vector

$$D = \begin{bmatrix} \lambda(S_1) \\ \vdots \\ \lambda(S_k) \end{bmatrix}.$$

Since $\mathrm{span}\{\mu(S_1),\ldots,\mu(S_k)\} \supset R$ it follows that $\mu(S) = (\mu_1(S),\ldots,\mu_n(S)) = \sigma M$, where $\sigma = (\sigma_1,\ldots,\sigma_k)$ is the vector of coefficients for $\mu(S)$ in terms of $\mu(S_1),\ldots,\mu(S_k)$. We also have $D = Md$ for some $d = (d_1,\ldots,d_n)$ since the rank of M is k. By the preceding lemma we know that $\mu(S) = \sigma M$ implies $\lambda(S) = \sigma D = \sigma Md = \langle \sigma M, d \rangle = \langle \mu(S), d \rangle$, as claimed.

Q.E.D.

We are now in a position to state and prove the theorem characterizing the cores of non-atomic linear production games, showing that for these games the only measures in the core are those that are given by the shadow prices of the initial resources. The proof is similar to Owen's, though the results here--in the non-atomic case--are stronger. Before stating the theorem, we will establish some notation.

Let $y^1,\ldots,y^L$ be the extreme points of $\{Ay \geq c,\ y \geq 0\}$ at which the optimal solution to (5.24) for $S = I$ is attained, i.e., those extreme points of $\{Ay \geq c,\ y \geq 0\}$ for which $v(I) = \langle y, \mu(I) \rangle$. We will denote by Y their convex hull: $Y = \mathrm{conv}\{y^1,\ldots,y^L\}$; Y is thus the set of all optimal solutions to the dual of (5.21) for $S = I$.

We shall also need the following

<u>Lemma 5.34</u>: Suppose R, the range of μ, has dimension k. Then there exist $S_1,\ldots,S_k \in C$ such that $\mathrm{span}\{\mu(S_1),\ldots,\mu(S_k)\} \supset R$ and $\mu(I)$ is a positive linear combination of $\mu(S_1),\ldots,\mu(S_k)$.

<u>Proof</u>: Let us first show that $\frac{1}{2}\mu(I)$ belongs to the relative interior of R, the range of μ. Since we are talking about the relative interior, we may assume, w.l.o.g., that $\dim R = n$, and talk about the interior instead.

Assume $\frac{1}{2}\mu(I)$ is not an interior point of R; then it is a boundary point. Since R is convex and compact, there is a hyperplane H, containing $\frac{1}{2}\mu(I)$, such that all of R lies on one side of H. Since $\frac{1}{2}\mu(I) = \frac{1}{2}\mu(\emptyset) + \frac{1}{2}\mu(I)$, and H is a hyperplane, the whole line $[0,\mu(I)]$ must be contained in H, that is the diagonal is contained in H. (That also means that H is a subspace of dimension $n-1$ in $\mathbb{R}^n$.) Since R has full dimension, there exists a set $S \in \mathcal{C}$ such that $\mu(S)$ belongs to the interior of R. That means that $\mu(S)$ is on that side of H where R is. But, since R is the range of a vector measure it always includes, for any point $x \in R$, the symmetric reflection of x with respect to $\frac{1}{2}\mu(I)$. So, the symmetric reflection of $\mu(S)$ through $\frac{1}{2}\mu(I)$ belongs to R as well, but this symmetric reflection lies on the other side of H, a contradiction. Therefore, $\frac{1}{2}\mu(I)$ is an interior point of R.

Now, since R is convex, there is a simplex contained in R such that $\frac{1}{2}\mu(I)$ is an interior point of that simplex, i.e., there exist $S_1,\ldots,S_n \in \mathcal{C}$ such that $\frac{1}{2}\mu(I) = \sum_{j=1}^{n} \alpha_j\mu(S_j)$, with $\alpha_j > 0$ $\forall_j$ and $\sum_{j=1}^{n} \alpha_j = 1$. Now we are done since $\mu(I) = 2(\frac{1}{2}\mu(I)) = \sum_{j=1}^{n} 2\alpha_j\mu(S_j)$, and $2\alpha_j > 0$ $\forall_j$.

Q.E.D.

We can now state

<u>Theorem 5.35</u>: Let $f \circ \mu$ be a non-atomic linear production game and suppose λ belongs to the core of $f \circ \mu$. Then d of Lemma 5.33 such

that $\lambda(S) = \langle \mu(S), d \rangle \quad \forall\, S \in \mathcal{C}$ is an optimal solution to (5.24) for $S = I$. (That is, $d \in Y$.)

<u>Proof</u>: By the preceding lemma, we can find $S_1, \ldots, S_k \in \mathcal{C}$, where $k = \dim R$, such that $\operatorname{span}\{\mu(S_1), \ldots, \mu(S_k)\} \supset R$ and such that

$$\mu(I) = \sum_{j=1}^{k} \alpha_j \mu(S_j) \quad \text{with} \quad \alpha_j > 0 \quad \forall_j. \tag{5.36}$$

W.l.o.g., we can assume that $S_i \cap S_j = \emptyset$ for $i \neq j$; otherwise, we can use Shapley's Lemma (Lemma 2.24) to obtain disjoint sets with those properties.

By Lemma 5.32, (5.36) implies that

$$\lambda(I) = \sum_{j=1}^{k} \alpha_j \lambda(S_j). \tag{5.37}$$

Denote by q the vector

$$q = \begin{bmatrix} \alpha_1 \lambda(S_1) \\ \vdots \\ \alpha_k \lambda(S_k) \end{bmatrix},$$

and by B the matrix

$$B = \begin{bmatrix} \alpha_1 \mu(S_1) \\ \vdots \\ \alpha_k \mu(S_k) \end{bmatrix}.$$

Now consider the following system of inequalities:

$$Ay \geq c \tag{5.38}$$

(5.39) $$By \leq q$$

(5.40) $$y \leq 0.$$

Suppose this system has a solution y^*. Adding the inequalities in (5.39) we get

$$\left\langle \sum_{j=1}^{k} \alpha_j \mu(S_j),\ y^* \right\rangle \leq \sum_{j=1}^{k} q_j = \sum_{j=1}^{k} \alpha_j \lambda(S_j) = \lambda(I) = (f\circ\mu)(I),$$

or

(5.41) $$\langle \mu(I),\ y^* \rangle \leq v(I),$$

where $v = f\circ\mu$. But, $v(I)$ is the minimum of the left-hand side of (5.41) over all y satisfying (5.38) and (5.40). So, we must have equality holding in (5.41), that is $\langle \mu(I),\ y^* \rangle = v(I)$. Note that this means that $y^* \in Y$. We therefore conclude that we must have equality holding in all the inequalities of (5.39) implying that

(5.42) $$\langle \alpha_j \mu(S_j),\ y^* \rangle = \alpha_j \lambda(S_j) \quad \forall_j.$$

Since $\alpha_j > 0 \quad \forall_j$, we can divide (5.42) through by it and get

(5.43) $$\lambda(S_j) = \langle \mu(S_j),\ y^* \rangle \quad \forall_j,$$

where $y^* \in Y$. The theorem now follows from Lemma 5.32.

Now suppose that the system (5.38) - (5.40) has no solution. It

can be thought of as a linear program with the 0 objective function, to be minimized. Its dual is

$$\max(\langle c,x\rangle - \langle q,z\rangle)$$

$$\text{s.t. } xA - zB \leq 0 \tag{5.44}$$

$$x \geq 0,\ z \geq 0.$$

This program is clearly feasible--0 is feasible--and, being the dual of an infeasible program, must be unbounded. Thus there exist $x \geq 0$, $z \geq 0$ such that

$$\langle c,x\rangle - \langle q,z\rangle > 0 \quad \text{or}$$

$$\langle c,x\rangle > \langle q,z\rangle, \quad \text{and} \quad xA \leq zB.$$

We can obviously choose z such that $z_j \alpha_j \leq 1 \quad \forall_j$, by dividing both x and z by an appropriate positive integer if necessary. Consequently, we can find a coalition $S \in C$ such that $\mu(S) = zB$. So, we must have $v(S)$ equal at least $\langle c,x\rangle$ as $xA \leq zB = \mu(S)$, but

$$\lambda(S) = \langle z,q\rangle < \langle c,x\rangle \leq v(S),$$

contradicting the fact that λ belongs to the core of $v = f\circ\mu$.

Q.E.D.

5.3 Summary and Examples

We have introduced in this chapter non-atomic linear production games,

generalizing previous work [22]. As in the finite case we saw that these games always have non-empty cores, and that the shadow prices of the initial resources ("raw materials") generate measures in the core. However, in contrast to the finite case, where the core may contain other allocations, in the non-atomic case those allocations are the whole core. We were also able to establish the fact that when the dual program of a non-atomic linear production game has a unique solution for the whole player set, this unique element in the core coincides with the value of the game. This is not the case in the finite games, as the following examples show:

Example 5.45: Let us assume a 2-player game, with two initial resources and one final good, requiring one unit each of the two initial resources with a unit price. Let the initial bundles of players 1 and 2 be:

$$b^1 = (3,1), \quad b^2 = (0,1).$$

The linear production game is thus given by

$$\begin{aligned} &\max x \\ (5.46) \qquad &\text{s.t. } x \le b_1(S) \\ &\quad\;\; x \le b_2(S) \\ &\quad\;\; x \ge 0. \end{aligned}$$

This gives us $v(1) = 1$, $v(2) = 0$, $v(12) = 2$. The value of this game is $\phi_1 = 3/2$, $\phi_2 = 1/2$. The core is:

$$\text{Core}(v) = \{x \in \mathbb{R}^2 | x_1 \geq 1,\ x_2 \geq 0,\ x_1 + x_2 = 2\}.$$

The dual of (5.36) for $S = N = \{1,2\}$ is

$$\begin{aligned} &\min(3y_1 + 2y_2) \\ (5.47) \qquad &\text{s.t. } y_1 + y_2 \geq 1 \\ &y_1, y_2 \geq 0. \end{aligned}$$

There is a unique solution to the dual, namely $y^* = (0,1)$. However, even though y^* is unique, $\phi_i \neq \langle b^i, y^* \rangle$, $i = 1,2$. Notice though, that in spite of the fact that $\phi_i \neq \langle b^i, y^* \rangle$, we still have ϕ belonging to the core of v.

In the next example we shall show how, even when the dual solutions are not unique, we may still have $\phi_i \neq \langle b^i, y \rangle$ for any y an optimal dual solution.

Example 5.48: Let us have the same setup as in Example 5.35, with the only exception being that $b^1 = (2,1)$ instead of $(3,1)$. (5.36) still describes our game, and we again obtain $v(1) = 1$, $v(2) = 0$, $v(12) = 2$, and $\phi_1 = 3/2$, $\phi_2 = 1/2$, and $\text{Core}(v) = \{x \in \mathbb{R}^2 | x_1 \geq 1,\ x_2 \geq 0,\ x_1 + x_2 = 2\}$. Again, $\phi \in \text{Core}(v)$. The dual, (5.37); has to be slightly changed. It now is

$$\begin{aligned} &\min(2y_1 + 2y_2) \\ (5.49) \qquad &\text{s.t. } y_1 + y_2 \geq 1 \\ &y_1, y_2 \geq 0. \end{aligned}$$

Here the set of optimal solutions is $\{(y_1,y_2): y_1+y_2 = 1\} \overset{\text{def}}{\equiv} A$. However, if we solve the two linear equations

$$\phi_i = \langle b^i,y\rangle, \quad i = 1,2$$

we find the unique solution to be $y = (3/2, 0) \notin A$.

In the last example of this chapter we shall exhibit that ϕ need not indeed even be in the core, and we will do it in such a way that the game has a unique optimal dual solution for $S = N$, the grand coalition. We shall then generate an equivalent non-atomic linear production game and point out the differences.

<u>Example 5.50</u>: Let $N = \{1,2,3\}$, and let $b^1 = (1,4)$, $b^2 = (2,3)$, $b^3 = (3,2)$ and let the production scheme be the same as before, i.e., given by (5.16). Then we get: $v(1) = 1$, $v(2) = 2$, $v(3) = 2$, $v(12) = 3$, $v(13) = 4$, $v(23) = 5$, $v(123) = 6$. Core(v) = $\{(1,2,3)\}$. Denote $a = (1,2,3)$. $\phi_1 = 7/6$, $\phi_2 = 13/6$, $\phi_3 = 16/6$, and indeed $\phi \notin$ Core(v). The dual for this game, for the grand coalition, is given by

$$\begin{aligned} &\min(6y_1 + 9y_2) \\ \text{(5.51)} \qquad &\text{s.t. } y_1 + y_2 \geq 1 \\ &y_1, y_2 \geq 0. \end{aligned}$$

The unqiue optimal solution is $y^* = (1,0)$ and indeed $a_i = \langle b^i,y^*\rangle$, but $\phi_i \neq \langle b^i,y^*\rangle$. So we see that even when the optimal dual solution for the grand coalition is unique, and it generates the

whole core, the value need not coincide with the core allocation. However, if we turn this game into a non-atomic one, this phenomenon disappears.

Let $I = [0,15]$ and C = the Borel subsets of I. We shall have three "types" representing our three original players.

$$I_1 = [0,1] \cup [6,10]$$

$$I_2 = [1,3] \cup [10,13]$$

$$I_3 = [3,6] \cup [13,15].$$

Let the two measure for the initial resources be

$$\mu_1(S) = \lambda(S \cap [0,6])$$

$$\mu_2(S) = \lambda(S \cap [6,15])$$

where λ is the Lebesque measure. We thus find that

$$\mu(I_1) = (\mu_1(I_1), \mu_2(I_1)) = (1,4)$$

$$\mu(I_2) = (\mu_1(I_2), \mu_2(I_2)) = (2,3)$$

$$\mu(I_3) = (\mu_1(I_3), \mu_2(I_3)) = (3,2),$$

and indeed the types correspond to the original players. The non-atomic linear production game is then given by

$$v(S) = f(\mu(S)) = \max x$$

$$\text{(5.52)} \qquad \text{s.t. } x \leq \mu_1(S)$$

$$x \leq \mu_2(S)$$

$$x \geq 0$$

or, the constraints can be written in matrix form--to conform with our previous notation by

$$xe \leq \mu(S)$$

where $e = (1,1)$. The dual of (5.41) for $S = I$ is

$$\min\langle\mu(I),y\rangle$$

$$\text{(5.53)} \qquad \text{s.t. } ye \geq 1$$

$$y \geq 0.$$

Numerically the objective function is $6y_1 + 9y_2$. The unique solution to (5.53) is $y^* = (1,0)$. We therefore conclude that $v(S) = \langle\mu(S),y^*\rangle = \mu_1(S)$ belongs to the core of v, and is the only member of the core, and it coincides with the value.

Finally, let us compute the values of the three types.

$$\nu(I_1) = \mu_1(I_1) = 1$$

$$\nu(I_2) = \mu_1(I_2) = 2$$

$$\nu(I_3) = \mu_1(I_3) = 3$$

and so we see that, in some sense, the only difference between the finite game and its non-atomic version is in the value assigned to each of the types. Note that none of them has the same value as in the finite case. This is a result, in a manner of speaking, of the freedom of "parts of a type" to interact with "parts of another type" in the non-atomic game, while in the finite game the players must interact in whole units.

We wish to remark further that the result that the core of a non-atomic linear production game consists only of measures that assign a monetary value to the initial resources owned by a coalition--using a set of shadow prices--is not too surprising. We know that this is the limit behaviour of finite linear production game when they are replicated, and in some sense the non-atomic version may be thought of as already having taken the replication process to its limit.

A remaining open question is what, if anything can be said about values of non-atomic linear production games for which the set of optimal dual solutions for the program for the whole player set consists of more than a single element. In [7], a partial answer was given for the asymptotic value (see [1] for definition and discussion of this notion). However, the value is defined here axiomatically, so different results may obtain.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The work is motivated by the following problem: bulk-service telephone lines were installed at Cornell University, to be used for long-distance calls. The charges paid to the telephone company are mostly fixed monthly charges and are not usage-related. The problem is how to allocate these costs back to the users in a per call fashion, and how to do it in a way that is fair and efficient.

next page

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The problem was solved by using the value of the associated non-atomic game. To be able to do this, the theory of non-atomic games had to be extended by weakening certain differentiability requirements. This is done here; in addition a number of results about full-range game are obtained.

Next the problem of non-atomic linear production games is studied. A number of results about the cores of such games are obtained, extending and strengthening similar results about finite linear production games. In addition, some results about the value of such games are established, and relationships between the core and the value are derived for a special case.

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